# Cubic Fourfolds, K3 surfaces and related topics

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# Introduction

A sentence of Daniel Huybrechts states that "Algebraic geometry starts with cubic polynomial equations. Everything of smaller degree, like linear maps or quadratic forms, belongs to the realm of linear algebra" ([Huy2]). Then it is not surprising that the algebraic geometers dedicated a huge amount of work to the study of cubic hypersurfaces.

The 1-dimensional case is that of elliptic curves, which have been studied intensively since 19-th century. This is true in particular for their group law and their moduli spaces, which brought to the development of the theory of modular forms and modular curves, with many applications both in Geometry and Arithmetic.

Smooth cubic surfaces are all rational and birational to a projective plane blown up at 6 points in a sufficiently general position. It is known that all of them contain 27 lines. The moduli space is a rational variety of dimension 4. The configuration of 27 lines gives rise to 36 pair of contractions of the same smooth cubic surface to a projective plane. Each of these pair is known as a double-six of lines of the cubic. The moduli space of these doubles-six configurations is still a rational variety, it is actually an open set of the weighted projective space $\mathbb{P}(1,2,3,4,5)$. A good reference for these results is [Dol2].

Smooth cubic threefolds are all unirational but not rational as well, as proved by Clemens and Griffiths in [CS]. This is a case of a variety which is reconstructed from its Hodge structure, that is a Torelli theorem holds for cubic threefolds. An equivalent property is that two cubic threefolds are isomorphic if and only if their intermediate Jacobians are. These are special principally polarized abelian varieties of dimension 5.

The cubic fourfolds are the subject of this thesis. As well as for cubic threefolds, the Hodge structure is not trivial and there exists a Torelli Theorem for cubic fourfolds, see [Vo]. The property of this case is that the Hodge structure is quite similar to that of a K3 surface. A reason explaining this fact is that, as shown by Beauville and Donagi in [BD], the Fano variety of lines of a cubic fourfold is a hyperkähler fourfold, deformation equivalent to the Hilbert scheme of subschemes of length 2 of a K3 surface.

The chapters from 1 to 4 are summaries of known results: in particular chapter 1 is a review of the theory of K3 surfaces, chapter 2 describes the various constructions of the moduli space of cubic fourfolds, chapter 3 recalls the notion of Hassett divisors and associated K3 surfaces, chapter 4 describes more in detail what is known about Pfaffian cubic fourfolds.

Chapter 5 contains our new results on the moduli of nodal and Pfaffian cubic fourfolds, namely a first geometric description of the intersection of the Hassett divisors $\mathcal{C}_6$ and $\mathcal{C}_{14}$. We show that this space contains two irreducible components characterized as follows. One component parametrizes the birational classes of nodal cubic fourfolds containing a cone over a quartic rational normal curve. The second one parametrizes birational classes of nodal cubic fourfolds containing a cone over a quintic elliptic curve. The corresponding associated K3 surfaces are described and the rationality of the first component is proved.

Chapter 6 contains the proof of the rationality of the universal K3 surface of genus 8 . The main idea is to use the Hassett correspondence in order to prove that the universal K3 surface of genus 8 is birational to a suitable GIT quotient of the linear system of cubic fourfolds containing a fixed rational normal scroll. We show that this quotient is naturally a projective bundle over a GIT quotient of the linear system of $(3,2)$-divisors in $\mathbb{P}^1 \times \mathbb{P}^1$.

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# Chapter 1

# K3 surfaces and their moduli

## 1.1 Basic definitions and properties

### 1.1.1 Complex K3 surfaces

**Definition 1.1.1.** A complex K3 surface is a compact connected manifold $X$ of dimension 2 such that

$$\Omega_X^2 \cong \mathcal{O}_X \text{ and } h^0(\Omega_X^1) = 0.$$

**Remark 1.1.2.** In other words the canonical sheaf of the complex manifold $X$ is trivial and its irregularity $q(X) = h^0(\Omega_X^1)$ is zero. Then the Hodge decomposition implies that the first Betti number $b_1(X)$ is zero. Since $b_1(X)$ is even a well known theorem of Siu implies that $X$ is a Kähler variety.

Let $D^r \subset D^a \subset D^n \subset \mathrm{Div}X$ respectively be the subgroups of divisors which are linearly , algebraically, numerically equivalent to zero. Then we have natural surjective homomorphisms

$$\mathrm{Pic}(X) \twoheadrightarrow \mathrm{NS}(X) \twoheadrightarrow \mathrm{Num}(X).$$

Indeed we just have by definition:

$$\mathrm{Pic}(X) \cong \mathrm{Div}(X)/D^r \text{ , } \mathrm{NS}(X) \cong \mathrm{Div}(X)/D^a \text{ , } \mathrm{Num}(X) \cong \mathrm{Div}(X)/D^n.$$

In what follows we mention, or sometimes revisit with ad hoc arguments, some further basic properties of a K3 surface.

**Lemma 1.1.3.** *Let $X$ be a complex manifold then $H^1(X, \mathbb{Z})$ has no torsion.*

*Proof.* This indeed follows from the exponential exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_B \overset{exp}{\to} \mathcal{O}_B^* \to 0.$$

Then, passing to its associated long exact sequence, we have

$$0 \to \mathbb{Z} \to \mathbb{C} \overset{exp}{\to} \mathbb{C}^* \to 0 \to \mathrm{H}^1(X, \mathbb{Z}) \to \dots.$$

Hence it follows that $\mathrm{H}^1(X, \mathbb{Z})$ injects in $\mathrm{H}^1(\mathcal{O}_X)$. Since the latter is a complex vector space, $\mathrm{H}^1(X, \mathbb{Z})$ has no torsion. □

**Proposition 1.1.4.** *Let $X$ be a K3 surface then its Betti number are as follows:*

$$b_0(X) = b_4(X) = 1\ ,\ \ b_1(X) = b_3(X) = 0\ ,\ b_2(X) = 22.$$

*Proof.* Poincarè duality implies the former equalities. The latter one follows by Noether's formula $12\chi(\mathcal{O}_S) = c_1^2(S) + c_2(S)$ for a smooth complex surface $S$. For $X = S$ we have $c_1^2(X) = 0$. Hence the Euler characteristic $c_2(X)$ of $X$ is 24, since $p_g(X) = h^0(\mathcal{O}_X) = 1$. This implies $b_2(X) = 22$. □

**Proposition 1.1.5.** *A K3 surface $X$ is simply connected.*

*Proof.* Let $t \in \pi_1(X)$ be a non zero element. Since $b_1(X) = 0$ then $t$ is a torsion element. Hence it defines a non ramified covering $\pi_t : \tilde{X} \to X$ of degree $n$, where $\tilde{X}$ is a complex surface with trivial canonical sheaf. $t$ acts on $\mathrm{H}^0(\Omega^1_{\tilde{X}})$ has an automorphism of order $n$. Assume the latter space is non zero, then there exists a non zero invariant holomorphic 1-form $\tilde{\omega}$. Since this descends on $X$ we have a contradiction. Hence we necessarily have $h^0(\Omega^1_{\tilde{X}}) = 0$ and, by definition, $\tilde{X}$ is a K3 surface. But then $c_2(\tilde{X}) = nc_2(X) = 24$ and hence $n = 1$, that is $t = 0$ in $\pi_1(X)$. □

**Proposition 1.1.6.** *The integral cohomology ring of a K3 surface has no torsion.*

*Proof.* By Poincarè duality and the universal coefficients theorem we have $\mathrm{H}_1(X, \mathbb{Z}) \cong \mathrm{H}^3(X, \mathbb{Z})$. Therefore it suffices to show that $\mathrm{H}_1(X, \mathbb{Z})$ and $\mathrm{H}^2(X, \mathbb{Z})$ have no torsion. The vanishing of $\mathrm{H}_1(X, \mathbb{Z})$ follows because this is the abelianisation of the fundamental group of $X$, which is zero. To prove that $\mathrm{H}^2(X, \mathbb{Z})$ has no torsion consider as above the exponential sequence of $X$ and its associated long exact sequence. We have in particular

$$0 \to \mathrm{H}^1(X, \mathcal{O}_X^*) \to \mathrm{H}^2(X, \mathbb{Z}) \overset{f}{\to} \mathrm{H}^2(X, \mathcal{O}_X) \to \dots$$

Let $t \in \mathrm{H}^2(X, \mathbb{Z})$ be a torsion element, then $f(t) = 0$, since it is a torsion element in $\mathrm{H}^2(X, \mathcal{O}_X) \cong \mathbb{C}$. This implies that $t \in \mathrm{H}^1(\mathcal{O}_X^*) \cong \mathrm{Pic}\,\mathrm{X}$. But then $t$ defines a finite unramified covering $\pi_t : \tilde{X} \to X$. Since $X$ is simply connected, $\pi_t$ is trivial and $t = 0$. □

Let $X$ be a K3 surface. The previous properties immediately imply that

$$\mathrm{Pic}(X) \cong \mathrm{NS}(X).$$

### 1.1.2 Structure of the cohomology

Using the previous properties and applying Poincaré duality, the Betti number of a K3 surface are the following:

$$b_0(X) = b_4(X) = 1\ ,\ b_2(X) = 22\ ,\ b_1(X) = b_3(X) = 0.$$

Moreover Poincaré duality implies that for a K3 surface $X$ the map

$$\lambda : \mathrm{H}^2(X, \mathbb{Z}) \times \mathrm{H}^2(X, \mathbb{Z}) \to \mathrm{H}^4(X, \mathbb{Z})$$

defined via cup product is a perfect pairing. Moreover the next property is very well known as well, see [Huy1].

**Proposition 1.1.7.** *$\lambda$ defines a structure of even, unimodular lattice on $\mathrm{H}^2(X, \mathbb{Z})$.*

Applying the well known classification, of these lattices up to isometries, it follows that

$$\mathrm{H}^2(X, \mathbb{Z}) \cong U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}.$$

where $U$ is a hyperbolic plane and $E_8(-1)$ is the exceptional lattice $E_8$ twisted by $-1$.

**Definition 1.1.8.** $U^{\oplus 3} \oplus E_8^{\oplus 2}$ is the K3 lattice.

**Proposition 1.1.9.** *The signature of the corresponding quadratic form on $\mathrm{H}^2(X, \mathbb{R})$ is $(3, 19)$ and it is $(1, 19)$ on the subspace $\mathrm{H}^{1,1}(X, \mathbb{R}) = \mathrm{H}^{1,1}(X, \mathbb{C}) \cap \mathrm{H}^2(X, \mathbb{R})$.*

### 1.1.3 Algebraic K3 surfaces

**Definition 1.1.10.** A complex K3 surface is said to be projective if there exists an ample class in $\mathrm{H}^{1,1}(X, \mathbb{Z})$.

For every smooth projective complex variety, algebraic and numerical equivalence coincide up to torsion for divisors. In particular, since the integral cohomology of a K3 surface $X$ has no torsion, we have:

**Proposition 1.1.11.** *Let $X$ be a complex projective K3 surface then*

$$\mathrm{Pic}(X) \cong \mathrm{NS}(X) \cong \mathrm{Num}(X).$$

**Example 1.1.12.** Some basic families of projective K3 surfaces are obtained considering smooth complete intersections in a projective space.as follows. Applying Lefschetz theorem to a complete intersection $X$ in $\mathbb{P}^n$, with $n \geq 3$, it follows $b_1(X) = 0$. Hence $h^0(\Omega^1_X) = 0$. Finally the standard application of adjunction formula implies that a smooth $X$ is a surface with trivial canonical class exactly in the following cases.

1. $X$ is a quartic surface in $\mathbb{P}^3$;
2. $X$ is complete intersection of a quadric and a cubic in $\mathbb{P}^4$;
3. $X$ is a complete intersections of three quadrics in $\mathbb{P}^5$.

**Example 1.1.13.** In the same way let $X \subset \mathbb{P}^n \times \mathbb{P}^m$ be a smooth complete intersection of $m + n - 2$ divisors of bidegrees $(a_i, b_i)$, $i = 1 \dots m + n - 2$. Assuming $a_i b_i \geq 2$ one obtains with the same arguments the following list.

1. $X$ is a surface of bidegree $(3, 2)$ in $\mathbb{P}^2 \times \mathbb{P}^1$;
2. $X$ is a surface of bidegree $(2, 1)$, $(2, 1)$ in $\mathbb{P}^3 \times \mathbb{P}^1$;
3. $X$ is a complete intersection of type $(2, 1)$, $(1, 2)$ in $\mathbb{P}^2 \times \mathbb{P}^2$.

Assume $X$ is a smooth complete intersection in a 3-dimensional product of projective spaces, then the only case not considered above is:
$X$ is a divisor of type $(2, 2, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$;

Finally let $X$ be a smooth quadratic section in $\mathbb{P}^g$ of a cone $V$ of vertex a point over a rational normal scroll $S$ of degree $g - 1$ in $\mathbb{P}^{g-2}$, see [GH] for the definition. Then the projection map of center the vertex $v$ of this cone defines a double covering

$$\pi_v : X \to S.$$

Applying once more the same arguments to the standard desingularization $V \times \mathbb{P}^1$ and the strict transform of $X$ on it, it follows that $X$ is a K3 surface of degree $2g - 2$ embedded in $\mathbb{P}^g$. This example is important because of the follwing theorem.

**Theorem 1.1.14.** *[[Huy1, ch.5 prop. 3.1]] Consider the Hilbert scheme $Hilb_X$ of $X$ in $\mathbb{P}^g$. Then, at the parameter point $x$ of $X$, one has:*

1. *$Hilb_X$ is smooth and irreducible at $x$ of dimension* $19 + \dim \operatorname{Aut} \mathbb{P}^g$,
2. *for a general $Y$ in the same component* $\operatorname{Pic}(Y)$ *is generated by* $\mathcal{O}_Y(1)$.

The theorem puts in evidence a well known property of projective K3 surfaces. For each $g \geq 3$ one has a family as above of degree $2g - 2$ K3 surfaces. This is further reconsidered in the next section.

## 1.2 Families of algebraic K3 surfaces

### 1.2.1 Polarized K3 surfaces

**Definition 1.2.1.** *A polarized K3 surface is a pair $(X, L)$ such that $X$ is a K3 surface and $L$ is an ample line bundle on $X$.*

This is equivalent to say that $X$ is endowed with an embedding in a projective space, provided by a suitable positive power $L^{\otimes m}$ of $L$. Let $X$ be embedded in $\mathbb{P}^N$, obviously $\mathcal{O}_X(1)$ is ample, with $m = 1$, and $(X, \mathcal{O}_X(1))$ is an example of polarized K3 surface. Recall that for K3 surfaces a stronger version of Fujita conjecture holds:

**Theorem 1.2.2** ([Huy1, ch.2 thm 2.7])**.** *Let $L$ be an ample line bundle on a K3 surface, then $L^{\otimes k}$ is globally generated for $k \geq 2$, very ample for $k \geq 3$.*

A similar result holds also for big and nef line bundles on K3 surfaces.

**Theorem 1.2.3** ([Huy1, che.2 them 3.4])**.** *Let $L$ be a big and nef line bundle, then $L^{\otimes k}$ is globally generated for $k \geq 2$.*

In what follows it will be natural to use pairs $(X, L)$ such that $L$ is big and nef. Following some use we adopt therefore the following definition.

**Definition 1.2.4.** *A pseudo-polarized K3 surface is a pair $(X, L)$ such that $X$ is a K3 surface and $L$ is a big and nef line bundle on $X$.*

Recall that a line bundle $L \in \mathrm{Pic}X$ is called *indivisible* or *primitive* if the quotient of $\mathrm{Pic}(X)$ by the group generated by it has no torsion.

Actually, in many cases to be considered, we will deal with an indivisible, big and nef line bundle $L$ such that $L$ itself is globally generated. The next proposition states conditions for which a line bundle on a K3 surface is globally generated or very ample.

**Proposition 1.2.5.** *Let $L$ be a big and nef line bundle on a K3 surface. If $L \cdot L \geq 4$ and $L \cdot E \geq 2$ for all curves $E$, then $L$ is globally generated. Furthermore if $L$ is ample, $L \cdot L \geq 4$ and $L \cdot E \geq 3$ for all curves $E$, then $L$ is very ample.*

**Definition 1.2.6.** *A primitively (pseudo) polarized K3 surface is a pair $(X, L)$, where $X$ is a K3 surface, $L$ an indivisible , (big and nef), ample line bundle.*

The value $2d = (L)^2$ is called *degree* of the polarization. Since the arithmetic genus of any curve in $|L|$ is $d + 1$ we will say, as usual, that $(X, L)$ has genus $d + 1$ or that $X$ is polarized in genus $g = d + 1$.

### 1.2.2 Projective models of K3 surfaces

We have seen that, under the assumptions ofthe latter proposition, a K3 surface (pseudo)-polarized by $L$ the associated rational map $\phi_L : X \to \mathbb{P}^g$ is birational onto its image $\phi_L(X)$. This is a surface of degree $2g-2$, where $g$ is the geometric genus of the general element of $|L|$. A natural question is the existence of such a K3 for any genus $g \geq 3$ and their properties. We have seen a concrete, well known, answer in the example 1.1.12. The example considered proves the existence. However it is a special case, where $\mathrm{Pic}(X)$ has rank two, of the following more general theory.

**Theorem 1.2.7** ([Huy1, 4.2]). *For $g \geq 3$ there exists a K3 surface of degree $2g-2$ in $\mathbb{P}^g$ primitively embedded and of Picard number one.*

A natural task is to know the equations defining a projective model $\phi_L(X)$ of a polarized K3 surface. For $g \geq 5$ the general K3 of genus $g$, embedded in $\mathbb{P}^g$, is defined by quadratic equations.

**Theorem 1.2.8** ([Sai, 7.2]). *Let $S$ be a K3 surface and $L$ be a polarization of degree at least 8, such that the general element of $|L|$ is non-hyperelliptic. Then $\varphi_L(S)$ is generated by homogeneous polynomials of degree 2 and 3. Furthermore it is generated by homogeneous polynomials of degree 2 except in these cases:*

- *There exists an irreducible curve $E \subset S$ such that $p_a(E) = 1$ and $E \cdot L = 3$;*
- *$L \cong \mathcal{O}_S(2B + \Gamma)$, where $B \subset S$ is an irreducible curve of genus 2, $\Gamma$ is a rational curve such that $\Gamma \cdot B = 1$.*

The same results have been extended to pseudo-polarized K3 surfaces. The map $\phi_L$ is a birational morphism onto its image under the same assumptions. Then the image $\phi_L(X)$ has finitely many rational double points as its only singularities.

### 1.2.3 Mukai constructions

In [Mu1] Mukai gave explicit descriptions of general K3 surfaces of genus $6 \leq g \leq 10$ and Picard number one in $\mathbb{P}^g$. Differently from the case of genus $g \leq 5$, for $g \geq 6$ these K3 surfaces are not complete intersections of $g-2$ hypersurfaces of $\mathbb{P}^g$. Mukai proved that, for those values of $g$, the general K3 surface is complete intersection in a certain homogeneous space $X$. Actually the latter is the quotient of a simply connected Lie group $G$ by a maximal parabolic subgroup $P$. $X$ is embedded in a projective space through the linear system $|\mathcal{O}_X(1)|$, where $\mathcal{O}_X(1)$ denotes the ample generator of the Picard group of $X$, whose rank is one. The group $G$ is the following, depending on $g$:

1. $g = 6, G = \mathrm{SL}(5), \dim(X) = 6, h^0(\mathcal{O}_X(1)) = 10$;
2. $g = 7, G = \mathrm{Spin}(10) \dim(X) = 10, h^0(\mathcal{O}_X(1)) = 16$;
3. $g = 8, G = \mathrm{SL}(6), \dim(X) = 8, h^0(\mathcal{O}_X(1)) = 15$;
4. $g = 9, G = \mathrm{Sp}(3), \dim(X) = 6, h^0(\mathcal{O}_X(1)) = 14$;
5. $g = 10, G =$ exceptional of type $G_2, \dim(X) = 5, h^0(\mathcal{O}_X(1)) = 14$.

In the case $g = 6$ the general K3 surface is the intersection of $X = G(2,5)$ with 3 hyperplanes and one quadric hypersurface. For $7 \leq g \leq 10$, the general K3 surface of genus $g$ is intersection of $X$ with $\dim(X) - 2$ hyperplanes. In particular for $g = 8$, the general K3 is intersection of $G(2,6)$ with 6 hyperplanes. This property is also known as Mukai linear section theorem.

## 1.3 Moduli spaces of K3 surfaces

The period domain is a fundamental tool to classify Hodge structures of complex K3 surfaces modulo isometries. The Torelli theorem, as given by Piateski-Shapiro and Shafarevich, ensures that this classification is equivalent to parametrize biholomorphic classes of complex K3 surfaces.

### 1.3.1 The period map

Denote by $\Lambda$ the K3 lattice and by $q$ its quadratic form. No lattice isomorphism $\mathrm{H}^2(X,\mathbb{Z}) \cong \Lambda$ is canonically defined a priori, even modulo the group of isometries of $\Lambda$.

**Definition 1.3.1.** A marked K3 surface is a pair $(X, \phi)$, where $X$ is a complex K3 surface and

$$\phi : \mathrm{H}^2(X,\mathbb{Z}) \to \Lambda$$

is a lattice isomorphism called marking. A morphism of marked K3 surfaces

$$f : (X, \phi) \to (X', \phi')$$

is a morphism $f : X \to X'$ such that $\phi \circ f^* = \phi'$.

Given a marked K3 surface $(X, \phi)$, we consider the one-dimensional subspace $\phi(\mathrm{H}^{2,0}(X)) \subset \Lambda \otimes \mathbb{C}$. This defines a point in $\mathbb{P}(\Lambda \otimes \mathbb{C})$. For $\sigma \in \mathrm{H}^0(X, \Omega^2_X) \setminus \{0\}$ we have that $\sigma^2 = 0$ and, by Hodge-Riemann bilinear relations, that $\sigma \wedge \bar{\sigma} > 0$ (see for example [GH, ch.0 sec. 7]). This means that the image of $\sigma$ is contained in an open subset of a quadric hypersurface defined by $q$, where $q$ denotes the quadratic form associated to the bilinear form on $\Lambda$.

**Definition 1.3.2.** The period domain is defined as follows:

$$\mathcal{D} := \{x \in \mathbb{P}(\Lambda \otimes \mathbb{C}) | q(x,x) = 0, \;\; q(x,\bar{x}) > 0\}.$$

The set of isomorphism classes of marked K3 surfaces is denoted by $\mathcal{N}$. The period map is defined as

$$\begin{aligned} \mathcal{N} &\xrightarrow{\mathcal{P}} \mathcal{D} \\ (X,\phi) &\longmapsto \phi(\mathrm{H}^{2,0}(X)). \end{aligned}$$

The image of $(X,\phi)$ is called the period of $(X,\phi)$.

We want to give a topological and complex structure to $\mathcal{N}$. In order to do this, we study the behaviour of the periods under deformations. Let $(X,\phi)$ be a marked K3 surface and

$$\pi : \mathcal{X} \to B$$

the universal deformation of $X$. Since $X$ is a regular surface we have, by Serre duality, $\mathrm{H}^2(X,\mathcal{T}_X) = H^0(X,\Omega^1_X) = 0$. This implies that the open neighborhood $B$ is smooth and we can assume that it is a polydisk, see [KNS] for a reference. The dimension of the basis is $\dim(\mathrm{H}^1(X,\mathcal{T}_X)) = h^{1,1}(X) = 20$. The sheaf $R^2\pi_*\mathbb{Z}$ is trivial and it has fibre $\mathrm{H}^2(X,\mathbb{Z})$.

This allows to identify canonically $\mathrm{H}^2(X_t,\mathbb{Z})$ to $\mathrm{H}^2(X,\mathbb{Z})$ for all $t \in B$. In this way $R^2\pi_*\mathcal{O}_{\mathcal{X}/B} \hookrightarrow \Lambda \otimes_{\mathbb{Z}} \mathcal{O}_B$ is a sub-bundle of rank one (since $h^{2,0}(X) = 1$). Thus we have a morphism from $B$ to $\mathbb{P}(\Lambda \otimes \mathbb{C})$. From what we discussed above, the image of this morphism is contained in $\mathcal{D}$. So there is a well defined map

$$\phi : B \to \mathcal{D}.$$

On the other side, since $\mathcal{X}$ is a family of marked surfaces, we also have a map

$$f : B \to \mathcal{N}$$

which satisfies the condition $\phi = \mathcal{P} \circ f$.

### 1.3.2 Torelli theorem and surjectivity

**Theorem 1.3.3** (Local Torelli Theorem)**.** *The differential of the local period map*

$$d\phi_t T_t B \to T_{\phi(t)}\mathcal{D}$$

*is an isomorphism. So $\phi$ is a local isomorphism.*

In particular, if we shrink $B$ enough, the map $f$ is injective. As a consequence we finally choose as a basis, for the topology we want to define on $\mathcal{N}$, the collections of all such maps $f$. This defines complex structure on $\mathcal{N}$: the transition functions must be holomorphic since they are induced by the maps between the bases of universal deformations.

**Definition 1.3.4.** *The complex manifold $\mathcal{N}$ is called the moduli space of marked K3 surfaces.*

$\mathcal{N}$ is not an irreducible manifold: it can be proved that it has two connected components.

**Theorem 1.3.5.** *(Surjectivity of the global period map). Let $\mathcal{N}^\circ$ be a connected component of $\mathcal{N}$. Then the restriction of the period map*

$$\mathcal{P} : \mathcal{N}^\circ \to \mathcal{D}$$

*is surjective.*

$\mathcal{N}$ does not behave well topologically, since it is not Hausdorff.

**Definition 1.3.6.** *Two points $x, y \in \mathcal{N}$ are called equivalent if for any two open neighborhoods $U \ni x, V \ni y$ we have $U \cap V \neq \emptyset$. In this case we write $x \sim y$.*

**Definition 1.3.7.** *The quotient space $\bar{\mathcal{N}} := \mathcal{N}/\sim$ is a Hausdorff space called the Hausdorff reduction of $\mathcal{N}$. We denote by $\rho : \mathcal{N} \to \bar{\mathcal{N}}$ the quotient map and by $\bar{\mathcal{P}}$ the map defined by the condition $\mathcal{P} = \bar{\mathcal{P}} \circ \rho$.*

**Proposition 1.3.8.** *$\bar{\mathcal{N}}$ is a Hausdorf complex manifold and $\bar{\mathcal{P}}$ is a topological covering. The same is true for the restriction to any connected component.*

It can be proved that $\mathcal{D}$ is simply connected. From this it follows the statement below.

**Proposition 1.3.9.** *Let $\bar{\mathcal{N}}^\circ$ a connected component of $\bar{\mathcal{N}}$. Then $\bar{\mathcal{P}} : \bar{\mathcal{N}}^\circ \to \mathcal{D}$ is an isomorphism.*

**Theorem 1.3.10** (Classical form of Global Torelli)**.** *Two K3 surfaces $X$ and $X'$ are isomorphic if and only if the Hodge structures $\mathrm{H}^2(X, \mathbb{Z})$ and $\mathrm{H}^2(X', \mathbb{Z})$ are isomorphic.*

### 1.3.3 Lattice polarized K3 surfaces

Lattice-polarized K3 surfaces are K3 surfaces whose Picard group contains a primitive embedding of a fixed abstract lattice, necessarily even.

**Definition 1.3.11.**
*Let $M$ be any abstract lattice of signature $(1, r-1)$. An $M$-polarized K3 surface is a pair $(S, \phi)$, where $S$ is a K3 surface and $\phi : M \to \mathrm{Pic}(S)$ is a fixed primitive embedding of lattices.*
*Two $M$-polarized lattice polarized K3 surfaces $(S, \phi)$ and $(S', \phi')$ are isomorphic if there exist an isomorphism of algebraic surfaces $f : S \to S'$ such that*

$$f^* \phi' = \phi.$$

Let $N = M^{\perp}$ be the orthogonal lattice in the K3 lattice. Denote by $N_{\mathbb{C}} := N \otimes \mathbb{C}$ and set

$$\mathcal{D}_M = \{\mathbb{C}v \in \mathbb{P}(N_{\mathbb{C}}) : v^2 = 0, v \cdot \bar{v} > 0\}.$$

Let $\Gamma_M$ the sub-lattice of the K3 lattice corresponding to isometries acting trivially on $M$. Then $\Gamma_M$ acts on $\mathcal{D}_M$. It is possible, using an analogous deformation argument, to define a period map

$$p_{\pi} : T \to \Gamma_M \backslash \mathcal{D}_M,$$

where $\pi : \mathcal{X} \to T$ is a family of $M$-polarized K3 surfaces.

**Theorem 1.3.12** (Torelli theorem for lattice polarized K3 surfaces)**.** *Let $(S, \phi)$ and $(S', \pi')$ be two $M$-polarized K3 surfaces and $p$ the period map. Then*

$$(S, \phi) \cong (S', \phi') \Longleftrightarrow p(S, \phi) = p(S', \phi').$$

If follows that $\Gamma_M \backslash \mathcal{D}_M$ is a coarse moduli space of lattice-polarized K3 surfaces.

## 1.4 Moduli of K3 surfaces of genus $g$

### 1.4.1 The construction

The construction of the moduli space of primitively polarized K3 surfaces can be done in a similar way. For any genus $g$ one can consider the set $N_g$ of triples $(X, L, \varphi)$, where $S$ is a K3 surface, $L$ a big and nef line bundle of degree $2g-2$ and $\varphi : \mathrm{H}^2(X, \mathbb{Z}) \to \Lambda$ a fixed isomorphism with the K3 lattice, which maps $L$ to the distinguished class $h_g = e_1 + df_1$.

Let $\Lambda_g = h_g^\perp$ and $\tilde{\mathrm{O}}(\Lambda_g)$ be the set of automorphisms of $\Lambda$ fixing $h_g$. Then $\tilde{\mathrm{O}}(\Lambda_g)$ acts on $N_g$ and $\tilde{\mathrm{O}}(\Lambda_g)\backslash N_g$ parametrizes couples $(S, L)$.

In a very similar way one can define the period map

$$\mathcal{P}_g : N_g \to \mathcal{D}_g, \;\; \overline{\mathcal{P}}_g : \tilde{\mathrm{O}}(\Lambda_g)\backslash N_g \to \tilde{\mathrm{O}}(\Lambda_g)\backslash \mathcal{D}_g$$

**Theorem 1.4.1.** *The maps $\mathcal{P}_g$ and $\bar{\mathcal{P}}_g$ are open embeddings.*

Differently from the non-polarized case, the right side of $\bar{\mathcal{P}}_d$ is a quasi-projective variety. It turns out that it is isomorphic to the space of primitively quasi polarized K3 surfaces:

$$\mathcal{F}_g \cong \tilde{\mathrm{O}}(\Lambda_g)\backslash \mathcal{D}_g.$$

### 1.4.2 Noether-Lefschetz divisors

The general primitively quasi polarized K3 surface, with a fixed genus g, has Picard lattice of rank 1, generated by the hyperplane section. The condition of having Picard lattice of rank at least 2 is divisorial in $\mathcal{F}_g$. For each $M$ primitive sublattice of $\Lambda$ containing $h_g$, there is an associated subset of $\mathcal{D}$:

$$S_M := \{z \in \mathcal{D} : z \cdot m = 0, \text{ for any } m \in M\}.$$

Denote by $\mathfrak{L}_{g,d,b} := \{M = h_g\mathbb{Z} \oplus \beta\mathbb{Z} : h_g \cdot \beta = d, \; \beta^2 = 2b\}$.

**Theorem 1.4.2.** *The set*

$$\mathcal{NL}_{g,d,b} := \Gamma_g \backslash \bigcup_{M \in \mathfrak{L}_{g,d,b}} S_M$$

*is an irreducible divisor of $\Gamma_g\backslash\mathcal{D}_g$, called Noether-Lefschetz divisor.*

In other terms, any curve in a very general K3 surface of degree $2g-2$ in $\mathbb{P}^g$ is cut by a hypersurface. The ones having Picard lattice of rank $\geq 2$ are union of countably many irreducible divisors.

### 1.4.3 GIT description for the initial values of $g$

For the initial values of $g$ there exist alternative characterizations of $\mathcal{F}_g$ as GIT quotients. For the values $g = 3, 4, 5$ the K3 surfaces are complete intersections, so they are easily described as GIT quotients.

1. $\mathcal{F}_3 \cong \mathbb{P}(\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^3}(4))) /\!\!/ \mathrm{PGL}(4)$;
2. $\mathcal{F}_4 \cong \mathcal{H}^{2,3}_{\mathbb{P}^4} /\!\!/ \mathrm{PGL}(5)$, where $\mathcal{H}^{2,3}_{\mathbb{P}^4}$ denotes the Hilbert scheme of complete intersections of a quadric and a cubic hypersurface in $\mathbb{P}^4$;
3. $\mathcal{F}_5 \cong \mathcal{H}^{2,2,2}_{\mathbb{P}^5} /\!\!/ \mathrm{PGL}(6)$, where $\mathcal{H}^{2,2,2}_{\mathbb{P}^5}$ denotes the Hilbert scheme of complete intersections of three quadric hypersurfaces in $\mathbb{P}^5$.

Mukai in [Mu1] described the K3 of genus $7 \leq g \leq 10$ as quotients of rational homogeneous spaces.

**Theorem 1.4.3** ([Mu1, 0.3]). *For each $g = 6, 7, 8, 9, 10$, denote by $G$ and $X$ following the notation of section 1.2.3 and $\bar{G} = G/Z(G)$. Then*

$$\mathcal{F}_g \cong G(n-2, \mathrm{H}^0(\mathcal{O}_X(1)))/\bar{G}.$$

### 1.4.4 Birational geometry of $\mathcal{F}_g$

A topic which received a wide attention is the birational classification of $\mathcal{F}_g$. The unirationality is clear for $g = 3, 4, 5$ since $\mathcal{F}_g$ is described as moduli of certain complete intersections. An immediate consequence of 1.4.3 is the following.

**Theorem 1.4.4** ([Mu1, 0.5]). *$\mathcal{F}_g$ is unirational for every $6 \leq g \leq 10$.*

Mukai also proved the unirationality of $\mathcal{F}_g$ for other particular values of $g$.

**Theorem 1.4.5** (Mukai). *The following moduli spaces of K3 surfaces are unirational:*

- $\mathcal{F}_{13}$ *([Mu4, 1.1]);*
- $\mathcal{F}_{18}$ *([Mu3, 0.3]);*
- $\mathcal{F}_{20}$ *([Mu3, 0.5]).*

Gritsenko, Hulek and Sankaran proved in [GHS, 1] proved that $\mathcal{F}_d$ is of general type for big values of $g$.

**Theorem 1.4.6.** *[GHS, 1] The moduli space $\mathcal{F}_{2d}$ of K3 surfaces with a polarisation of genus $g$ is of general type for any $g > 62$ and for $g = 47, 51, 55, 58, 59$ and 61.*

*If $g \geq 41$ and $g \neq 42, 45, 46$ or 48 then the Kodaira dimension of $\mathcal{F}_g$ is non-negative.*

### 1.4.5 The universal pointed K3 surface $\mathcal{F}_{g,n}$: what is known?

Since the automorphism group of the generic K3 surface is trivial, the Universal Hilbert scheme of dimension 0 and length $n$ and the universal $n$-pointed K3 surface are well defined.

**Definition 1.4.7.** $\mathcal{F}_g^{[n]}$ is the moduli space of triples $(S, L, Y)$, where $S$ is a K3 surface, $L$ is primitive big and nef line bundle, $Y \subset S$ is a 0-dimensional subscheme of length $n$.

$\mathcal{F}_{g,n}$ is the moduli space of triples $(S, L, Y)$, where $S$ is a K3 surface, $L$ is primitive big and nef line bundle, $Y \subset S$ is an ordered set of $n$ points of $S$.

The rationality results for $\mathcal{F}_g$ obtained by Mukai can be extended to $\mathcal{F}_g^{[n]}$ and $\mathcal{F}_{g,n}$ as pointed out by Farkas and Verra in [FV1].

**Theorem 1.4.8** ([FV1, 5.1])**.**

*(i)* $\mathcal{F}_{g,g+1}$ *is unirational for* $g \leq 10$.

*(ii)* $\mathcal{F}_{11,1}$ *is unirational. The Kodaira dimension of* $\mathcal{F}_{11,11}$ *and* $\mathcal{F}_{11}^{[11]}$ *equals 19.*

Also the universal K3 of genus 13 is unirational, since Mukai proved in [Mu3] that it is dominated by a rational homogeneous space. Notice by the way that $\mathcal{F}_{g,1}$ cannot be of general type. The answer is negative, as remarked in [FV1, 5.4], because $\mathcal{F}_{g,1} \to \mathcal{F}_g$ is fibred in Calabi-Yau varieties, that is the K3 surfaces of the universal family. Then, the so called Iitaka's easy addition formula implies that $\kappa(\mathcal{F}_{g,1}) \leq \dim(\mathcal{F}_g) = 19$.

An interesting question is to know for what values of $g$ and $n$ $\mathcal{F}_{g,n}$ and $\mathcal{F}_g^{[n]}$ are near to being rational. By this we mean that the variety considered is at least uniruled, with possibly additional properties culminating in the property of being rational. For instance Farkas and Verra proved that the universal K3 surface of degree 14 is rational by using the connection between K3 surfaces and cubic fourfolds.

**Theorem 1.4.9.** *[FV1, 1,2,1.3] The universal K3 surface* $\mathcal{F}_{14,1}$ *is birational to a* $\mathbb{P}^{12}$*-bundle over the moduli space* $\mathfrak{h}_{\mathrm{scr}}$ *of 3-nodal septic scrolls of* $\mathbb{P}^5$*.*
*Furthermore* $\mathfrak{h}_{\mathrm{scr}}$ *is birational to* $\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3 /\!/ \mathfrak{S}_3$ *and is thus rational.*

*The author of this manuscript proved in [DiT] that* $\mathcal{F}_{8,1}$ *is a rational variety as well. The proof is part of the work of this thesis and the details are in chapter 6.*

# Chapter 2

# Cubic fourfolds and their moduli

## 2.1 Basic definitions and properties

### 2.1.1 Cubic fourfolds and GIT-stability

The moduli space of cubic fourfolds parametrizes cubic hypersurfaces of $\mathbb{P}^5$ up to projective automorphisms. So it is a GIT quotient

$$\mathbb{P}(\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^5}(3))) /\!\!/ \mathrm{PGL}(6).$$

The dimension of this space is $\dim(\mathbb{P}(\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^5}(3)))) - \dim(\mathrm{PGL}(6)) = 20$. As is well known from Geometric Invariant Theory, in order to have a so called good quotient, one needs to determine the semistable locus cubic hypersurfaces under the action of PGL(6). Radu Laza in [Laz1] exhaustively answered this question in the case of cubic fourfolds.

**Theorem 2.1.1** ([Laz1, 1.1]). *A cubic fourfold $X$ is not stable if and only if the following conditions holds:*

*(i) $X$ is singular along a curve $C$ spanning a linear subspace of dimension at most 3 of $\mathbb{P}^5$;*

*(ii) $X$ contains a singularity that deforms to a singularity of class $\tilde{E}_r$ (for $r = 6, 7, 8$).*

*In particular a cubic fourfold containing only isolated simple singularities (of type $A_n$ for $n \geq 1$, $D_n$ for $n \geq 4$, $E_n$ for $n = 6, 7, 8$) are stable.*

The above theorem implies that it makes sense to discuss about the space of cubic fourfolds with at most simple isolated singularities, which will be denoted by $\mathcal{C}$. Laza described the stability conditions also for cubic fourfolds having a higher dimensional singular locus.

**Proposition 2.1.2.** *[Laz1, 6.5] Let X be a cubic fourfold with* $\dim(\mathrm{Sing}(X)) \geq 2$*. Then one of the following holds:*

*(i) the singular locus contains a plane or a quadric surface;*

*(ii) the singular locus S is a cone over a rational normal quartic curve, in which case X is the secant variety of this cone;*

*(iii) the singular locus S is a Veronese surface in* $\mathbb{P}^5$*, in which case X is its secant variety.*

*Furthermore X is semistable only in case (iii).*

The other main result of Laza was the construction of a compactification of $\mathcal{C}$.

**Theorem 2.1.3** ([Laz1, 1.2])**.** *The moduli space* $\mathcal{C}$ *of cubic fourfolds having at worst simple singularities is compactified by the GIT quotient* $\overline{\mathcal{C}}$ *by adding six irreducible boundary components* $\gamma_1, ..., \gamma_6$*. A semistable cubic fourfold X with minimal orbit corresponding to a generic point in a boundary component has one of the following geometric properties:*

$\gamma_1$ : *X is singular along a line and a quartic elliptic curve;*

$\gamma_2$ : *X has two* $\tilde{E}_8$ *singularities of the same modulus;*

$\gamma_3$ : *X is singular along a conic and has an isolated* $\tilde{E}_7$ *singularity;*

$\gamma_4$ : *X has three* $\tilde{E}_6$ *singularities.*

$\gamma_5$ : *X is singular along a rational normal curve of degree* 4*; X is stable;*

$\gamma_6$ : *X is singular along a sextic elliptic curve; X is stable.*

*Furthermore, the boundary components* $\gamma_2$ *and* $\gamma_5$ *are 3-dimensional and they meet along a surface* $\sigma$*. Moreover* $\sigma$ *meets the 2-dimensional boundary components* $\gamma_3$ *and* $\gamma_6$ *along a curve* $\tau$*.* $\tau$ *meets the 1-dimensional components* $\gamma_1$ *and* $\gamma_4$ *in a single point* $\zeta$*.*

### 2.1.2 Structure of the cohomology

The structure of the lattice $\mathrm{H}^4(X, \mathbb{Z})$ of a cubic fourfold plays a central role in the theory that will follow.

Convention In what follows we often adopt the notation

**L**

for a fixed abstract lattice isometric to $\mathrm{H}^4(X, \mathbb{Z})$.

We will always denote by $h$ the hyperplane class in $H^2(X, \mathbb{Z})$. Therefore $h^2$ is the cohomology class of a 2-dimensional linear section of $X$ and

$$(h^2)^2 = 3.$$

**Definition 2.1.4.** *We say that* $\mathbf{L}$ *is the cubic fourfold lattice.*

The canonical bundle of a smooth cubic fourfold $X \subset \mathbb{P}^5$ is $\mathcal{O}_X(-3)$. The non zero Betti numbers, are the following:

$$b_0(X) = b_2(X) = b_6(X) = b_8(X) = 1, b_4(X) = 23$$

and the Hodge diamond is

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 0 & & 0 & & & \\ & & 0 & & 1 & & 0 & & \\ & 0 & & 0 & & 0 & & 0 & \\ 0 & & 1 & & 21 & & 1 & & 0. \end{array}$$

The lattice structure of the middle cohomology of a cubic fourfold $X$ is

$$\mathrm{H}^4(X, \mathbb{Z}) \cong (+1)^{\oplus 21} \oplus (-1)^{\oplus 2} := \mathbf{L}.$$

The Hodge-Riemann bilinear relations imply that $\mathrm{H}^4(X, \mathbb{R})$ has signature $(21, 2)$. The lattice is unimodular by Poincaré duality and $(h^2)^2 = 3$. This characterizes $\mathbf{L}$ in the classfication of unimodular lattices.

The primitive cohomology is the orthogonal lattice in $H^4(X, \mathbb{Z})$ of the squared hyperplane class:

$$\mathrm{H}^4(X, \mathbb{Z})_{\mathrm{prim}} = (h^2)^{\perp}.$$

Differently from the full cohomology, the primitive middle cohomology is an even unimodular lattice of signature $(20, 2)$. One has

$$\mathrm{H}^4(X, \mathbb{Z})_{\mathrm{prim}} \cong A_2 \oplus U^{\oplus 2} \oplus E_8^{\oplus 2}.$$

For what it follows it is now useful to introduce the Fano variety of lines of $X$ and its cohomology.

## 2.2 The Fano variety of lines $F(X)$

### 2.2.1 Basic definitions and properties

Given a projective hypersurface $X \subset \mathbb{P}^n$ of degree $d$, its Fano scheme of m-planes contained in $X$ is set theoretically defined as

$$F(X, m) = \{\ell \in \mathbb{G}(m, n) : \ell \subset X\}.$$

As a scheme $F(X, m)$ is appropriately defined as the fibre, at the element $X \in |\mathcal{O}_{\mathbb{P}^n}(d)|$, of the natural projection morphism

$$v : \mathbb{I} \to |\mathcal{O}_{\mathbb{P}^n}(d)|.$$

where $\mathbb{I}$ is the incidence correspondence

$$\mathbb{I} := \{(X, L) \in |\mathcal{O}_{\mathbb{P}^n}(d)| \times \mathbb{G}(m, n)| \ L \subset X\}.$$

At least if the hypersurface $X$ is smooth, it turns out that $F(X, m)$ is the Hilbert scheme of $L$ in $X$. Very good references on the Fano scheme of projective hypersurfaces are [BV] and [AK].

In the case of cubic hypersurfaces, the lecture notes of Huybrechts ([Huy2]) are reccomended since they represent an exhaustive and strongly simplified updating of the entire theory.

A natural task is understanding the geometric properties of $F(X, m)$.

**Theorem 2.2.1** ([Huy2, 1.5])**.** *If for an arbitrary hypersurface $X \subset \mathbb{P}^n$ of degree $d$ the Fano variety $F(X, m)$ is not empty, then*

$$\dim(F(X, m)) \geq n(m+1) - \binom{m+d}{d}.$$

*Moreover the equality holds for generic $X \in |\mathcal{O}_{\mathbb{P}^n}(d)|$ unless $F(X, m)$ is empty.*

The case of our interest is $d = 3$ and $m = 1$. In this case one has

$$\dim(F(X, 1)) \geq 2n - 6.$$

Another relevant case is $d = 3, m = 2$: in this situation we have

$$\dim(F(X, 2)) \geq 3n - 16.$$

In particular the right side is non-negative only for $n \geq 6$. This implies that the general cubic fourfold does not contain planes.

Convention In the case of lines, that is $m = 1$, we denote the Fano scheme of $X$ just by $F(X)$. With some abuse we will also say that $F(X)$ is the Fano variety (of lines) of $X$.

This name originates from the investigations, due to the algebraic geometer Gino Fano (1871-1952), on $F(X)$ when $X$ is a cubic hypersurface in $\mathbb{P}^4$. The variety $F(X)$ is a smooth surface for each smooth cubic $X \subset \mathbb{P}^4$.

The smoothness of an algebraic variety is equivalent to the property that its tangent sheaf is locally free. This remark is somehow of relevant interest when studying Hilbert schemes, where the structure of the tangent bundle to the Hilbert scheme $H_W$, of a given (smooth) subscheme $W$ of a smooth variety $V$, is explicitely related to the normal bundle of $W$ in $V$.

A fundamental property of Grothendieck's theory of the Hilbert scheme implies that, for a point of it corresponding to a (smooth) subscheme $W \subset V$ the space of global sections of the normal bundle $N_{W|V}$ is the Zariski tangent space to the Hilbert scheme itself. In our situation this implies the next proposition, see [Huy2, Ch.3, 1.9].

**Proposition 2.2.2.** *Let $L \subset X$ be an $m$-plane contained in a variety $X \subset \mathbb{P}^n$ which is smooth along $L$. Then the tangent space $T_L F(X,m)$ of $F(X,m)$ is naturally isomorphic to* $\mathrm{H}^0(L, \mathcal{N}_{L/X})$.

If $X$ is a cubic hypersurface and $L$ is a line in $X - \mathrm{Sing}(X)$, there is a precise characterization of $\mathcal{N}_{L/X}$. At first we recall that, as a vector bundle on $\mathbb{P}^1$, $\mathcal{N}_{L/X}$ splits as follows, by a well known theorem of Grothendieck.

$$\mathcal{N}_{L/X} \cong \mathcal{O}_L(a_1) \oplus ... \oplus \mathcal{O}_L(a_{n-2})$$

**Lemma 2.2.3** ([Huy2, Ch.3, 1.12])**.** *Let $L$ and $X$ be as above, then*

$$\mathcal{N}_{L/X} \cong \mathcal{O}_L(a_1) \oplus ... \oplus \mathcal{O}_L(a_{n-2}),$$

*with*

$$(a_1, ..., a_{n-2}) \in \{(1, ..., 1, 0, 0), (1, ..., 1, -1)\}.$$

**Corollary 2.2.4** ([Huy2, Ch.3, 1.15])**.** *If $L \subset X$ is a line in a smooth cubic hypersurface, then*

$$\mathcal{T}_X|_L \cong \mathcal{O}_L(2) \oplus \mathcal{N}_{L/X}.$$

Note that in both the cases the restricted tangent bundle has dimension $2n-6$, so its stalks have that dimension.

For any projective variety, we define its universal line as:

$$\mathbb{L} := \{(\ell, x) \in F(X) \times X : x \in \ell\}.$$

**Proposition 2.2.5.** [Huy2, Ch.3, 1.16] *Let $X$ be a smooth cubic hypersurface, $\mathcal{N}_{\mathbb{L}/F(X)}$ the normal bundle of the inclusion $\mathbb{L} \subset F(X) \times X$ and $p : \mathbb{L} \to F(X)$ the first projection. Then*

$$\mathcal{T}_{F(X)} \cong p_* \mathcal{N}_{\mathbb{L}/F(X) \times X}.$$

For smooth cubic hypersurfaces of dimension $\geq 2$ there always exists lines contained in it .

**Proposition 2.2.6.** [Huy2, 1.17] *Let $X \subset \mathbb{P}^n$ a smooth cubic hypersurface, $n \geq 3$. Then $F(X)$ is a smooth projective variety of dimension $2n - 6$.*

### 2.2.2 The hyperkähler structure

A hyperkähler manifold is a compact simply connected manifold whose space of global holomorphic 2-forms is spanned by a everywhere non-degenerate sympletic form.

**Remark 2.2.7.** In particular a hyperkähler manifold is symplectic, then it is even dimensional.

**Example 2.2.8.** Complex K3 surfaces are hyperkähler manifolds.

Generalizing the example above, Beauville in [Bea] constructed a special family of hyperkähler manifolds.

**Proposition 2.2.9.** *[[Bea]] Let $S$ be a smooth K3 surface, $n > 1$. Then:*

*(i) $S^{[n]}$ is a hyperkähler variety;*

*(ii) There exists an isomorphism of weight-two Hodge structures compatible with the canonical integral forms*

$$\mathrm{H}^2(\mathrm{Hilb}^n(S), \mathbb{Z}) \cong \mathrm{H}^2(S, \mathbb{Z}) \oplus \mathbb{Z}\delta,$$

*where on the right hand side $\delta$ is a class of type (1, 1) and the integral form is the direct sum of the intersection pairing on $S$ and the integral form given by $\delta^2 = -2(n-1)$. In particular $b_2(S^{[n]}) = 23$*

The Fano variety of a cubic fourfold is a hyperkähler manifold of this type.

**Theorem 2.2.10.** *Let $X$ be a cubic fourfold, Then $F(X)$ is a hyperkähler variety. Its Hodge diamond is.*

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 1 & & 21 & & 1 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
1 & & 21 & & 23 & & 21 & & 1
\end{array}$$

### 2.2.3 The Abel-Jacobi map

The Abel-Jacobi map connects the cohomology of a cubic fourfold $X$ with the cohomology of its Fano variety $F(X)$.

**Definition 2.2.11.** Let $X$ be a cubic fourfold and $F$ be its Fano variety. Let

$$P := \{(l, x) : x \in l\} \subset F \times X$$

be the incidence variety. Let $\pi_1 : P \to F, \pi_2 : P \to X$ be the projections. The Abel-Jacobi map is the homomorphism of cohomology groups

$$\alpha := \pi_{1*}\pi_2^* : \mathrm{H}^4(X, \mathbb{Z}) \to \mathrm{H}^2(F, \mathbb{Z}).$$

Let $h \in \mathrm{H}^2(X, \mathbb{Z})$ be the hyperplane class of $X$ and $g \in \mathrm{H}^2(F, \mathbb{Z})$ the hyperplane class of $F \subset G(2, 6) \subset \mathbb{P}(\bigwedge^2 \mathbb{C}^6)$. Consider the family $F_S$ of lines in $X$ intersecting a codimension two linear section $S = \mathbb{P}^3 \cdot X$. This family is cut on $F(X)$ by the Schubert divisor of $G(2, 6)$ defined by $\mathbb{P}^3$. Since this has class $g$ and $S$ has class $h^2$, just the definition of $\alpha$ implies

$$\alpha(h^2) = g.$$

**Definition 2.2.12.** *We denote by* $\mathrm{H}^4(X, \mathbb{Z})_{\mathrm{prim}}$ *the subspace of* $\mathrm{H}^4(X, \mathbb{Z})$ *given by the condition* $x \cdot h^2 = 0$ *and by* $\mathrm{H}^2(F, \mathbb{Z})_{\mathrm{prim}}$ *the subspace of* $\mathrm{H}^2(F, \mathbb{Z})$ *given by the condition* $x \cdot g^3 = 0$.

**Remark 2.2.13.** $\mathrm{H}^4(X, \mathbb{Z})_{\mathrm{prim}}$ is the primitive cohomology, in the frame of Lefschetz theory. This motivates the notation.

A fundamental contribution to the understanding of the Fano variety of lines $F(X)$ of a smooth cubic fourfold is due to Beauville and Donagi. In [BD, 6] they show the following theorem.

**Theorem 2.2.14.** *For every $X$ the Fano variety $F(X)$ is a smooth deformation of the Hilbert scheme of* 2*-points of a K3 surface.*

More precisely they produce one example of smooth cubic fourfold with this property, which implies the same for any $X$. The example is reconsidered several times in the forthcoming part of this thesis, which is indeed dedicated to some properties of such a family of fourfolds $X$ and to the family of their one-nodal limits.

Here let us say in advance that the K3 surface $S$, considered by Beauville and Donagi, is a general smooth linear section of the Plücker embedding of the Grassmannian $G(2,6)$. Let $L = \mathcal{O}_S(1)$, we are therefore dealing with a polarized K3 surface $(S, L)$ of degree $d = 14$ and genus $g = 8$. It is also known, by the mentioned work of Mukai, that $(S, L)$ defines a general point of its moduli space $\mathcal{F}_8$. In particular $\mathrm{Pic}(S)$ is generated by $L$. Relying on the surface $S$ and on the example of $X$ they construct, Beauville and Donagi prove several important properties of $F(X)$ as follows.

**Proposition 2.2.15.** [BD, 6]

- *The Abel-Jacobi map induces an isomorphism* $\mathrm{H}^4(X,\mathbb{Z})_{\mathrm{prim}} \to \mathrm{H}^2(F,\mathbb{Z})_{\mathrm{prim}}$.
- *The quadratic form* $\phi_0$ *on* $\mathrm{H}^2(F,\mathbb{Z})_{\mathrm{prim}}$ *defined by*

$$\phi_0(u,v) := \frac{1}{6}g^2uv$$

  *is entire, even and of discriminant 3.*
- *For any* $x, y \in \mathrm{H}^4(X,\mathbb{Z})_{\mathrm{prim}}$

$$\phi_0(\alpha(x),\alpha(y)) = -x \cdot y.$$

*Proof.* We prove first the 3rd assertion. Let $x \in \mathrm{H}^4(X,\mathbb{Z})$. Since $\pi_1$ is a $\mathbb{P}^1$-fibration, there exist $x_1 \in \mathrm{H}^2(F,\mathbb{Z}), x_2 \in \mathrm{H}^4(F,\mathbb{Z})$ such that:

$$\pi_2^*x = (\pi_1^*x_1)\pi_2^*h - \pi_1^*x_2. \tag{2.1}$$

Applying $\pi_{1*}$ we get $x_1 = \alpha(x)$. in particular there is $g_2 \in \mathrm{H}^2(F,\mathbb{Z})$ such that:

$$\pi_2^*h^2 = (\pi_1^*g)\pi_2^*h - \pi_1^*g_2. \tag{2.2}$$

Suppose now that $x \cdot h = 0$. Multiplying 2.1 by $\pi_2^*h$ and putting together with 2.2 we get the relations

$$x_2 = x_1g, \qquad x_1g_2g_2 = 0. \tag{2.3}$$

Taking the square of 2.1, we find:

$$\pi_2^*x^2 = -\pi_1^*(gx_1^2)\cdot\pi_2^*h + \pi_1^*(g^2x_1^2),$$

then multiplying by $\pi_1^*g$:

$$6x^2 = -g^2\alpha(x)^2, \text{ then the 3rd assertion follows.}$$

We prove 2nd assertion. It is sufficient to prove it in the particular case when $F \cong S^{[2]}$: there is a canonical isomorphism

$$\mathrm{H}^2(F,\mathbb{Z}) = \mathrm{H}^2(S,\mathbb{Z}) \oplus \mathbb{Z}\delta$$

where $2\delta$ is the class of the exceptional divisor of $S^{[2]}$. Denote by $\phi$ the quadratic form on $\mathrm{H}^2(F,\mathbb{Z})$ such that $\phi(s+n\delta) = s^2 - 2n^2$, for $s \in \mathrm{H}^2(s,\mathbb{Z})$ and $n \in \mathbb{Z}$; let $l \in \mathrm{H}^2(s,\mathbb{Z})$ be the class of an hyperplane section. A standard calculation gives:

$$g = 2l - 5\delta,$$

and, for $y, v \in \mathrm{H}^2(F,\mathbb{Z})$,

$$g^3 u = 18\phi(g,u),$$

$$g^2 uv = 6\phi(u,v) + 2\phi(g,u)\phi(g,v).$$

So $\phi$ coincides with $\phi_0$ on $\mathrm{H}^2(F,\mathbb{Z})_{\mathrm{prim}}$. The second assertion follows. To prove the 1st assertion we compute the expression of $g_2$ in $\mathrm{H}^4(F,\mathbb{Z})$:

$$gg_2 u = 15\phi(g,u), \quad \text{for } u \in \mathrm{H}^2(F,\mathbb{Z}).$$

Considering 2.3, this implies that $\alpha(\mathrm{H}^4(X,\mathbb{Z})_{\mathrm{prim}}) \subset \mathrm{H}^2(F,\mathbb{Z})_{\mathrm{prim}}$. Hence $\alpha$ is an isometry of $\mathrm{H}^4(X,\mathbb{Z})$ with $(\mathrm{H}^2(F,\mathbb{Z})_{\mathrm{prim}}, -\phi_0)$. Since both of them have discriminant 3, then $\alpha$ is an isomorphism. □

## 2.3 Moduli of cubic fourfolds via periods

### 2.3.1 Period map and Torelli theorem

Relying on the previous description of the Hodge theory of $X$, we now outline Claire Voisin's proof of Torelli theorem for cubic fourfolds.

*As usual $\mathcal{C}^\circ$ will denote the moduli space of smooth cubic fourfolds.*

For any smooth cubic fourfold $X$, there exists an isometry

$$\phi : \mathrm{H}^4(X,\mathbb{Z}) \to \mathbf{L}$$

such that $\phi(h^2) = \eta$, where $\mathbf{L}$ denotes the abstract *cubic fourfold lattice* fixed in the previous section. We recall that $\mathbf{L}$ is a unimodular odd lattice of signature $(21,2)$ for which there exists $\eta \in L$ such that $(\eta,\eta) = 3$ and for which $L_{\mathrm{prim}} := \eta^\perp$ is an even lattice. Then,

**Definition 2.3.1.** *Let* $\mathbf{L}$ *and* $\phi$ *as above,* $\phi$ *is called a marking of* $X$*.*

For any marking $\phi$, we can identify $\mathrm{H}^4(X,\mathbb{C})_{\mathrm{prim}}$ with $\mathbf{L}_{\mathrm{prim},\mathbb{C}} := \mathbf{L}_{\mathrm{prim}} \otimes_{\mathbb{Z}} \mathbb{C}$.

From Hodge theory $\mathrm{H}^{3,1}(X)$ is a distinguished subspace of $\mathbf{L}_{\mathrm{prim},\mathbb{C}}$, which means:

- $\mathrm{H}^{3,1}(X)$ is isotropic: it is spanned by a form $\alpha$ such that $(\alpha, \alpha) = 0$;
- The hermitian form $-(\alpha, \bar{\beta})$ is positive on $\mathrm{H}^{3,1}(X)$.

Consider the quadric hypersurface $Q \subset \mathbb{P}(\mathbf{L}_{\mathrm{prim},\mathbb{C}})$ defined as

$$Q := \{[\alpha] \in \mathbb{P}(\mathbf{L}_{\mathrm{prim},\mathbb{C}}) : (\alpha, \alpha) = 0\}.$$

Let $U$ be the open (in the euclidean topology) subset of $Q$ defined as

$$U := \{[\alpha] \in Q : -(\alpha, \bar{\alpha}) > 0\}.$$

The real Lie group $\mathrm{SO}(\mathbf{L}_{\mathrm{prim},\mathbb{R}}) \cong \mathrm{SO}(20, 2)$ acts transitively on $U$. Moreover $\mathrm{SO}(\mathbf{L}_{\mathrm{prim},\mathbb{R}})$ has two components: one of them acts by exchanging $\mathrm{H}^{3,1}(X)$ and $\mathrm{H}^{1,3}(X)$. $U$ has two connected components which parametrize the subspaces $\mathrm{H}^{3,1}(X)$ and $\mathrm{H}^{1,3}(X)$. We denote the first one by $\mathcal{D}'$. It is a 20-dimensional open complex manifold, called *local period domain* for cubic fourfolds with markings. So the marking associates to $X$ an element of $\mathcal{D}$ (its *period point*). This is the classifying space for polarized Hodge structures arising from cubic fourfolds.

We denote by $\Gamma \subset \mathrm{Aut}(\mathrm{H}^4(X, \mathbb{Z}))$ the set of lattice automorphisms preserving the class $h^2$ and by $\Gamma^+ \subset \Gamma$ be the subgroup preserving $\mathcal{D}'$.

**Definition 2.3.2.** *The orbit space* $\mathcal{D} := \Gamma^+ \backslash \mathcal{D}'$ *is called global period domain.*

This is indeed a 20-dimensional quasi-projective variety. In fact in [Sat] it is shown that the manifold $\mathcal{D}'$ is a bounded symmetric domain of type IV. The group $\Gamma^+$ is arithmetically defined and acts holomorphically on $\mathcal{D}'$. In this situation we may introduce the Borel-Baily ([BB]) compactification compatible with the action of $\Gamma^+$, so that the quotient is projective. Moreover, $\mathcal{D}$ is a Zariski open subvariety of this quotient.

Observe that any cubic fourfold $X$ determines uniquely a point in $\mathcal{D}$.

**Definition 2.3.3.** The map $\mathcal{P} : \mathcal{C}^\circ \to \mathcal{P}(\mathcal{C}^\circ) \subset \mathcal{D}$ is called the period map.

By Hodge theory, this a holomorphic map of analytic spaces (actually it is algebraic). The main result in [Vo] is the following.

**Theorem 2.3.4** (Global Torelli)**.** *$\mathcal{P}$ is an isomorphism of complex analytic varieties.*

It turns out that it is a local isomorphism. So it is sufficient to show that it is injective. To show this fact Voisin used an alternative description of the central cohomology of a smooth cubic fourfold.

The idea of Voisin for her proof of Torelli theorem for $X$ was to:

1. how that $\mathcal{P}|_{\mathcal{H}}$ is injective, where $\mathcal{H}$ is the codimension 1 subvariety given by cubics containing a plane;
2. show that $\mathcal{P}^{-1}(\overline{\mathcal{P}(H)}) = \mathcal{H}$;
3. show that $\mathcal{P}$ is unramified in codimension 1.

Suppose that $X$ is a cubic fourfold containing a plane $\Pi$. Let

$$\Pi' := \{S \in \mathbb{G}(3,5) : \Pi \subset S\}$$

be the family of 3-dimensional projective subspaces containing $\Pi$. We have that $\Pi' \cong \mathbb{P}^2$. There is a natural projection map

$$\begin{aligned} \mathbb{P}^5 \setminus \Pi &\xrightarrow{\pi} \Pi' \\ P &\longmapsto \Pi \wedge P \end{aligned}$$

the induced morphism

$$\tilde{\pi} : \mathrm{Bl}_{\Pi} X \to \Pi'$$

is a quadric surface fibration with singular fibers over a sextic curve $C \subset \Pi'$. Let

$$f : F_1(X/\mathbb{P}^2) \to \mathbb{P}^2$$

be the relative variety of lines of the quadric fibration, i.e. $f^{-1}(p)$ is the Fano variety of lines of the quadric surface $\tilde{\pi}^{-1}(p)$. The Stein factorization of $f$ is

$$F_1(X/\mathbb{P}^2) \xrightarrow{q} S \xrightarrow{r} \mathbb{P}^2$$

where $q$ is a $\mathbb{P}^1$-bundle and $r$ is a double cover branched along $C$. Then $S$ is a degree 2 $K3$ surface which parametrizes the rulings of the fibres of $\tilde{\pi}$. The idea is to connect the cohomology of $S$ to the cohomology of $X$ and then to use the Torelli theorem for $K3$ surfaces.

Let $D \subset F(X)$ be the subvariety of $F(X)$ parametrizing the lines meeting $\Pi$. It can be described in the following way:

$$D := \pi_1(\{(l,s) \in F(X) \times S : l \in q^{-1}(s)\}).$$

Note that $q^{-1}(s)$ is the ruling of lines parametrized by $s$.

**Proposition 2.3.5.** *The restriction to $\pi_2(D)$ of the Abel-Jacobi map is an isomorphism of Hodge structures*

$$\alpha : \mathrm{H}^4(X,\mathbb{C}) \to \mathrm{H}^2(D,\mathbb{C}).$$

*Proof.* We prove the dual version of the theorem, showing that

$$\alpha^* : \mathrm{H}_2(D,\mathbb{C}) \to \mathrm{H}_4(X,\mathbb{C})$$

is an isomorphism. Let $Z_D \subset D \times X$ be the restriction of the incidence variety to $D$. The projection $\pi_2 : Z_D \to X$ is a morphism of degree 2. Let $x \in X - \Pi$ such that the quadric surface determined by $x$ is smooth, then there are exactly two lines of this quadric containing $x$. So the map $\pi_2^* : \mathrm{H}_4(Z_D,\mathbb{C}) \to \mathrm{H}_4(X,\mathbb{C})$ is surjective. Moreover,

$$\mathrm{H}_4(Z_D,\mathbb{C}) = p_1^*(H_2(D,\mathbb{C}) \oplus \mathrm{H}_4(D,\mathbb{C}))$$

and the map $\mathrm{H}_4(D,\mathbb{C}) \to \mathrm{H}_4(X,\mathbb{C})$ factors through $\mathrm{H}_4(Y,\mathbb{C})$, where $Y$ is a hyperplane section of $X$. Since the image of $\mathrm{H}_4(U,\mathbb{C})$ is contained in the image of $\mathrm{H}_2(D,\mathbb{C})$ we have that $\alpha^*$ is surjective. A dimension count shows that the two spaces have the same dimension, so $\alpha$ is a bijection. □

Since $D$ is a fibration over $S$ whose fibres are rational curves, we have that the Hodge structure of $D$ is identified to the one of $S$, then the period map on the space of cubics containing a plane can be identified with the period map on $\mathcal{F}_2$ (since $S$ has degree 2).

More precisely, let $\Pi, Q$ be the cohomology classes in $\mathrm{H}^4(X,\mathbb{Z})$ of a plane and of a quadric such that $\Pi + Q = h^2$. Denote by $W$ the image in $\mathrm{H}^2(D,\mathbb{Z})$ under the restriction of the Abel-Jacobi map to the orthogonal complement of $\mathrm{Span}(\Pi, Q)$.

**Proposition 2.3.6.** *The subspace $W$ is contained in $q^*\,\mathrm{H}^2(S,\mathbb{Z})_{\mathrm{prim}}$ and, for any $a, b$, orthogonal to $\Pi$ and $Q$, we have that*

$$(a,b)_X = -\langle \alpha(a), \alpha(b) \rangle.$$

In $\mathrm{H}^2(S,\mathbb{Z})$ there is a special class $k$ such that

$$W = \{a \in \mathrm{H}^2(S,\mathbb{Z})^0 : \langle a, k\rangle_S \equiv 0 (\mathrm{mod}\, 2)\}.$$

We can identify $k$ as a class in $\mathrm{H}^2(S,\mathbb{Z}/(2\mathbb{Z}))$ inducing morphisms in $\mathrm{Hom}(\mathrm{H}^2(S,\mathbb{Z})_{prim}, \mathbb{Z}/2\mathbb{Z})$.
One may compute in $\mathrm{NS}^2(X)$ the products:

$$\langle P, Q\rangle = 3, \ \ \langle Q, Q\rangle = -4, \ \ \langle P, Q\rangle = -2.$$

The idea of Voisin was that the lattice $\mathrm{H}^4(S,\mathbb{Z})$ is obtained from the orthogonal sum

$$\mathrm{Span}(P, Q) \oplus W.$$

For this, Voisin also needed that $W^*/W \cong \mathbb{Z}/8\mathbb{Z}$ and that the module $\mathrm{Span}(P, Q) \oplus W$ can be extended to a unimodular lattice with an integral bilinear form.

The last thing to be proved was the if two lattices $W, W'$ determine isomorphic Hodge structures on the cubic, then they induce an automorphism of $\mathrm{H}^2(S,\mathbb{Z})$ sending one to the other, while preserving the Hodge structure and the polarisation. All these facts can be summarized in the following statement.

**Theorem 2.3.7.** *The Hodge structure on S plus the choice of the lattice* $\mathrm{Span}(P, Q) \subset \mathrm{H}^4(S,\mathbb{Z})$ *determine S. Conversely the polarised Hodge structure of S plus the distinguished class k determine the Hodge structure of S.*

This defines a morphism from the moduli space of cubics containing a plane to $\mathcal{F}_2$.

### 2.3.2 Special Cubic Fourfolds

*Every smooth cubic fourfold* $X$ contains surfaces which are not complete intersections. A typical example is the intersection

$$S = X \cdot \mathbb{P}^1 \times \mathbb{P}^2,$$

where $\mathbb{P}^1 \times \mathbb{P}^2$ denotes the Segre embedding of such a product in $\mathbb{P}^5$. It is obvious that $S$ is a surface of degree 9 which is not a complete intersection. However the class of $S$ in $H_4(X,\mathbb{Z})$ is the class of a complete intersection. Indeed consider the scheme defined by $y = x^3 = 0$, where $y$ and $x$ are independent linear forms on $\mathbb{P}^5$. Then it is easy to see that the non reduced complete intersection $S_o = X \cdot \{y = x^3 = 0\}$ and $S$ are homologous in $X$.

Notice also that the class of $S_o$ in $H_4(X, \mathbb{Z})$ is $3h^2$, where $h \in H_6(X, \mathbb{Z})$ is the class of a hyperplane section. Hence we can certainly conlude this introductory remark and example by saying that:

*$S$ has the homology class of a complete intersection.*

After this remark, the natural question to be proposed is:

*Which are the families of smooth cubic fourfolds containing surfaces $S$ which are not homologous in $X$ to a complete intersection?*

We will see that, in the moduli of cubic fourfolds, there are countably many divisorial families of cubic fourfolds $X$ containing surfaces which in $X$ are not homologous to a complete intersection.

In the moduli space their union is the locus of classes of cubic fourfolds $X$ such that the Neron-Severi group $\mathrm{NS}^2(X)$, of codimension 2 algebraic cycles modulo numerical equivalence, has rank at least 2.

Let us preliminarly recall on this subject that the Hodge conjecture is true for smooth cubic hypersurfaces, so that

$$\mathrm{NS}^2(X) = \mathrm{H}^4(X, \mathbb{Z}) \cap \mathrm{H}^{3,1}(X)^{\perp}$$

This case of the Hodge conjecture is due to Zucker, [Zuc]. The Hodge-Riemann bilinear relations imply that $\mathrm{NS}^2(X)$ is positive definite.

**Definition 2.3.8** ([Has1, 3.1.1])**.** *A cubic fourfold $X$ is special if it contains an algebraic surface $T$ which is not homologous to a complete intersection.*

In particular $X$ is special if and only if and only if

$$\mathrm{NS}^2(X)_{\mathrm{prim}} := \mathrm{NS}^2(X) \cap \mathrm{H}^4(X, \mathbb{Z})_{\mathrm{prim}} \neq \{0\}$$

**Definition 2.3.9.** *Let $K$ be a positive definite lattice of rank two with a given distinguished element $h^2$ such that $(h^2)^2 = 3$. A $K$-marked special cubic fourfold is a pair $(X, \varphi)$, where*

$$\varphi : K \to \mathrm{NS}^2(X)$$

*is a primitive embedding preserving the image of $h^2$.*

A labelling of a $K$-marked special cubic fourfold is the the vector $\varphi(K)$. A special cubic fourfold is called typical if it has a unique labelling. The definition of speciality can be expressed in terms of Hodge structures. As above we denote by **L** our abstract cubic fourfold lattice.

**Definition 2.3.10.** *Let $\mathcal{D}' \subset \mathbb{P}(L_{\mathbb{C}})$ be the space parametrizing Hodge structures. For any $K \subset L$ primitive of rank 2 and containing $h^2$, we denote by*

$$\mathcal{D}'_K := \{x \in \mathcal{D}' : K_{\mathrm{prim}} \subset x^{\perp}\}.$$

*Here $K_{\mathrm{prim}} = K \cap \mathrm{NS}^2(X)_{\mathrm{prim}}$.*

### 2.3.3 The admissible divisorial families

The spaces $\mathcal{D}'_K$ are hyperplane sections of $\mathcal{D}'$ and in fact irreducible divisors. Let $\Gamma^+_K \leq \Gamma^+$ be the subgroup preserving the lattice $K$. Then the image of the natural map

$$\Gamma^+_K \backslash \mathcal{D}'_K \to \mathcal{D} := \Gamma^+ \backslash \mathcal{D}'$$

is an irreducible divisor in $\mathcal{D}$. It turns out that two distinct lattices $K_1$ and $K_2$ correspond to the same divisor of $\mathcal{D}$ if and only if there exists $\gamma \in \Gamma^+$ such that $\gamma(K_1) = K_2$.

An important property of special cubic fourfolds is that they are distributed in a countable family of irreducible divisors, more precisely:

Let $K \subset \mathrm{NS}^2(X)$ be a positive definite rank-two saturated sub-lattice containing $h^2$ and let $[K]$ be the $\Gamma^+$-orbit of $K$. Let $\mathcal{C}_{[K]}$ be the family of special cubic fourfolds $X$ such that $\mathrm{NS}^2(X) \supset K'$ for some $K' \in [K]$.

**Theorem 2.3.11** ([Has1, 3.1.4]). *Then $\mathcal{C}_{[K]}$ is an irreducible (possibly empty) algebraic divisor of $\mathcal{C}$. Every special cubic fourfold is contained in some such $\mathcal{C}_{[K]}$.*

The main steps of Hassett's proof of 2.3.11 are to:

1. Give the an Hodge-theoretical interpration of being special: $X$ is special if and only if $\mathrm{NS}^2(X)_{\mathrm{prim}} \cap \mathrm{H}^{3,1}(X, \mathbb{C})^\perp \neq \{0\}$.
2. Use the Hodge-theoretical interpretation to interpret each divisor as an hyperplane section of the period domain.
3. Interpret any divisor as a quotient of a bounded symmetric domain of type IV by an arithmetic group acting holomorphically. This allows to prove that the divisor is irreducible and algebraic.

Two saturated sub-lattices of $\mathrm{NS}^2(X)$ containing $h^2$ in the same $\Gamma^+$-orbit have obviously the same discriminant. The converse is true.

**Proposition 2.3.12** ([Has1, 3.2.4]). *Let $K, K'$ be saturated rank-two non-degenerate sub-lattices of $L$ containing $h^2$. Then $K = \gamma(K')$ for some $\gamma \in \Gamma^+$ if and only if $K$ and $K'$ have the same discriminant.*

The proposition above combined with theorem 2.3.11 implies that the irreducible divisorial families of special cubic fourfolds are labelled by $2 \times 2$ symmetric matrices with non negative integer entries.

**Definition 2.3.13** ([Has1, 3.2.1]). Let $X$ be a special cubic fourfold. A labelling of $X$ is a choice of saturated rank-two sublattice $K \subset \mathrm{NS}^2(X)$ containing $h^2$. The discriminant of $(X, K)$ is the determinant of the intersection matrix of $K$.

### 2.3.4 The Noether-Lefschetz divisors of $\mathcal{C}$

Not for all $d \in \mathbb{N}^+$ there exist labelled cubic fourfolds $(X, K)$ of discriminant $d$. Hassett shows in [Has1, 3.2.2] the following fundamental result

**Theorem 2.3.14** (Hassett Existence Theorem)**.** *Let $X$ be a stable cubic fourfold, then a labelled cubic fourfold $(X, K)$, with $K$ of discriminant $d$, exists if and only if*

$$d > 0 \text{ and } d \equiv 0, 2( \mod 6).$$

*In this case $X$ defines a point in $\mathcal{C}$ of an irreducible divisor $\mathcal{C}_d$ which is the closure of the isomorphism classes of cubic fourfolds labelled by the same $K$.*

[Has1, 3.2.2], [Has1, 4.3.1]. Every divisor in $\mathcal{C}$ which is labelled by some $K$ is often called a *Noether-Lefschetz divisor.* In other words a Noether-Lefschetz divisor of $\mathcal{C}$ is an irreducible divisor whose general point is defined by a cubic fourfold $X$ such that $NS(X) \cong K$ for some $K$ as above.

We discuss more on this theorem in the next chapter. In it a special sequence of effective divisors $\mathcal{C}_d$ is related to K3 surfaces. The union of the divisors of a suitable, proper sub-sequence of it is conjecturally the locus in $\mathcal{C}$, parametrizing rational cubic fourfolds.

### 2.3.5 The boundary of $\mathcal{C}^\circ$

A natural question is to know the complement of the locus of periods of smooth cubic fourfolds inside the period space $\mathcal{D}$. For the values $d = 2, 6$ it is known that the divisors $\mathcal{D}_2, \mathcal{D}_6$ are in this complement.

Anyway, as explained in section 4.2 of [Has1], the Torelli map can be extended to the divisor in $\mathcal{C}$ parametrizing cubic fourfolds with one ordinary singular point and its image is $\mathcal{D}_6$. More precisely there is a natural sextic $K3$ surface associated to a cubic fourfold with an ordinary double point. The extended Torelli map associates any $[X] \in \tilde{\mathcal{C}} \setminus \mathcal{C}$ to the limiting Hodge structure arising from the smoothing of $X$.

# Chapter 3

# Cubic fourfolds and K3 surfaces

## 3.1 Nodal cubic fourfolds

For this section we assume that $X$ is a cubic fourfold with an ordinary singular point $o$ and such that $\mathrm{Sing}(X) = \{o\}$.

### 3.1.1 The natural desingularization

### 3.1.2 The associated K3 surface $S$

The projection from $o$ gives a birational map

$$\pi_o|_X : X \dashrightarrow \mathbb{P}^4.$$

$C_o X \cap X$ is a cone over a sextic K3 surface $S = \pi_o(C_o X \cap X)$. So $S$ parametrizes the lines contained in $X$ passing thorugh $o$. It follows that $\pi_o|_X$ factors as

$$\begin{tikzcd}
\mathrm{Bl}_S \mathbb{P}^4 \arrow[r, "q_1"] \arrow[d, "q_2"'] & X \\
\mathbb{P}^4 &
\end{tikzcd}$$

The map $q_1$ is the blow-up of the double point $o$ and $q_2$ is the blow-down of the lines contained in $X$ passing through $o$.

This construction induces a birational map

$$\mathcal{C}_6 := \mathcal{C} \setminus \mathcal{C}^\circ \to \mathcal{F}_4.$$

### 3.1.3 $S[2]$ and $F(X)$

There is a birational correspondence

$$\begin{array}{ccc} S^{[2]} & \xrightarrow{\quad f \quad} & F(X) \\ \ell_1 + \ell_2 & \longmapsto & \ell_3 \end{array}$$

where $S$ is interpreted as the set of lines of $X$ passing through $o$ and

$$\ell_1 + \ell_2 + \ell_3 = \langle \ell_1, \ell_2 \rangle \cdot \mathrm{X}$$

Notice that $f$ is a morphism if no plane through $o$ is in $X$. Assuming this it is known that $F(X)$ is the 2-symmetric product of $S$, that is

$$F(X) = S \times S / < i >,$$

where $i$ is the involution exchanging the factors of $S \times S$. Moreover the morphism $f : S^{[2]} \to F(X)$ is the usual map whose inverse is the blowing up of the diagonal $\Delta_S$ of $S \times S$. In particular it turns out that

$$\mathrm{Sing}(F(X)) = \Delta_S.$$

## 3.2 Cubic fourfolds with an associated K3

### 3.2.1 The admissible Neron-Severi lattices $\mathbb{L}_d$

Not for all $d \in \mathbb{N}^+$, the divisor $\mathcal{D}_d$ is non-empty.

**Proposition 3.2.1** ([Has1, 3.2.2]). *Let $(X, K)$ be a labelled cubic fourfold of discriminant $d$. Then*

$$d > 0 \textit{ and } d \equiv 0, 2 (\mod 6).$$

### 3.2.2 The admissible divisors $\mathcal{C}_d$

We have seen that $\mathcal{D}_2$ correspond to limiting Hodge structure of cubic fourfolds singular along a Veronese surface.

$\mathcal{D}_6$ corresponds to limiting Hodge structures of cubic fourfolds having an ordinary double point, so there are no smooth cubic fourfolds in $\mathcal{C}_6$.

**Theorem 3.2.2** ([Has1, 4.3.1]). ***Existence of special cubic fourfolds]*** *Let $d > 6$ be an integer with $d \equiv 0, 2 (\mod 6)$. Then $\mathcal{C}_d \cap \mathcal{C}^\circ$ is non-empty.*

Hassett used a deformation argument to prove this fact: first he described special cubics in $\mathcal{C}_6$ admitting a labelling of discriminant $d$. Then he proved that there is a smoothing

$$\mathcal{X} \to B,$$

where $X_t$ is smooth for $t \neq 0$. In particular this proves that $\mathcal{C}^\circ \cap \mathcal{C}_d$ is non-empty for $t \neq 0$.

### 3.2.3 The Hassett correspondence $\mathcal{C}_d \to \mathcal{F}_g$, $g = \frac{1}{2}d + 1$

**Definition 3.2.3.** Let $(X, K_d)$ be the labelled special cubic fourfold of discriminant $d$. By definition, the orthogonal complement to $K_d$

$$K_d^\perp \subset \mathrm{H}^4(X, \mathbb{Z})^\perp$$

is the nonspecial cohomology lattice of $(X, K_d)$. $W_{X,K_d}$ denotes the polarized Hodge structure on $\mathrm{H}^4(X, \mathbb{C})_{\mathrm{prim}}$. This is called the nonspecial cohomology of $(X, K_d)$.

**Theorem 3.2.4** ([Has1, 5.2.1])**.** *Let $(S, K_d)$ be a labelled special cubic fourfold of discriminant $d$, with nonspecial cohomology $W_{X,K_d}$. There exists a polarized K3 surface $(S, L)$ such that*

$$W_{X,K_d} \cong \mathrm{H}^2(S, \mathbb{C})_{\mathrm{prim}}(-1)$$

*if and only if the following conditions are satisfied:*

1. $4 \nmid d$ *and* $9 \nmid d$*;*
2. $p \nmid d$ *if* $p$ *is an odd prime such that* $p \equiv 2(\mod 3)$.

We denote by $\Lambda_{\mathrm{prim},d}$ the lattice isomorphic to the middle cohomology of a K3 surface of degree $d$. The theorem asserts that there is an isomorphism of lattices $K_d^\perp \cong \Lambda_{\mathrm{prim},d}$. In this situation $W_{X,K_d}(+1)$ has the form of the primitive cohomology of a quasi polarized K3 surface. The Torelli theorem for K3 surfaces allows to reduce the theorem to the following proposition:

**Proposition 3.2.5** ([Has1, 5.2.2])**.** *Let $\Lambda_{\mathrm{prim},d}$ be the cohomology lattice of a degree $d$ K3 surface and let $K_d^\perp$ be the nonspecial cohomology of a labelled cubic fourfold of discriminant $d$. Then $K_d^\perp \cong \Lambda_{\mathrm{prim},d}$ if and only if the following conditions are satisfied:*

1. *$4 \nmid d$ and $9 \nmid d$;*

2. *$p \nmid d$ if $p$ is an odd prime, $p \equiv -1(\mod 3)$.*

These values of $d$ are called *admissible*.

Hassett makes explicit the relation between $\mathcal{F}_g$ and $C_d$, where $d = 2g - 2$ is an admissible value.

**Theorem 3.2.6.** *Let $d$ be an admissible value and $j_d : K_d^{\perp} \to -\Lambda_{\mathrm{prim},d}$ an isomorphism. Denote by $C_d^{\mathrm{mar}}$ the moduli space of marked special cubic fourfolds of discriminant $d$. If $d \neq 6$ there is an induced isomoprhism*

$$i_d : \mathcal{D}_d^{\mathrm{mar}} \to \mathcal{F}_g$$

*Furthermore $\mathcal{D}_6^{\mathrm{lab}} \to \mathcal{F}_4$.*

**Proposition 3.2.7.** *The natural map*

$$\mathcal{D}_d^{\mathrm{mar}} \to \mathcal{D}_d^{\mathrm{lab}}$$

*is an isomorphism of $d \equiv (2 \mod 6)$ and a double cover if $d \equiv (0 \mod 6)$. Furthermore $\mathcal{D}_d^{\mathrm{mar}}$ is connected for all admissible $d \neq 6$.*

From the previous results one can conclude that, for an admissible value of $d = 2g - 2$ and $d > 6$, Hodge theory defines a natural rational map

$$f : \mathcal{F}_g \to \mathcal{C}_d.$$

This has degree two if $d \equiv (0 \mod 6)$ and one if $d \equiv (2 \mod 6)$. For $d = 6$ has degree 1.

## 3.3 Examples and the rationality problems

In this section we describe some example of divisors of special cubic fourfold and a some results about rationality. For any surface $S$ contained in a smooth 4-dimensional smooth projective variety $X$ the equality $\langle S, S \rangle = c_2(\mathcal{N}_{S/X})$ holds. In particular if $X$ is a cubic fourfold we have that

$$\langle \sigma, \sigma \rangle = 6h^2 + 3hK_S + K_S^2 - \chi_S, \tag{3.1}$$

where $\sigma$ is the class of $S$ in $\mathrm{NS}^2(X)$ and $\chi_S$ is the Euler characteristic.

### 3.3.1 d=8: Cubic fourfolds containing a plane

Suppose that $X$ is a smooth cubic fourfold containing a plane $P$. Using formula 3.1 it can be easily proved that $[P]^2 = 3$ in $\mathrm{NS}^2(X)$. it follows that

$$K_8 = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix},$$

which has discriminant 8. So $\mathcal{C}_8 \subset \mathcal{C}$ can be parametrized as the locus of cubic fourfolds containing a plane. There are other possible characterizations, for example:

1. Cubic fourfolds containing quadric surfaces: by taking the residue intersection of $X$ with a $\mathbb{P}^3$ containing $P$;
2. Cubic fourfolds containing a quartic del Pezzo surfaces: the residue intersection of a quadric threefold containing a quadric $Q$ contained in $X$.

### 3.3.2 d=12: Cubic fourfolds containing a cubic scroll

Let $X$ be a smooth cubic fourfold containing a rational normal cubic scroll $R$. We have

$$\langle \rho, \rho \rangle = 7,$$

where $\rho$ is the cohomology class of $R$. Then the intersection matrix of $h^2$ and $\rho$ is

$$\begin{pmatrix} 3 & 3 \\ 3 & 7 \end{pmatrix},$$

whose determinant is 12. It can be proved that the closure of the locus of cubics containing such a scroll is an irreducible family in $\mathcal{C}$, whose closure is then $\mathcal{C}_{12}$.

### 3.3.3 d=14: Cubic fourfolds containing a quartic scroll

Let $X$ be a smooth cubic fourfold containing a rational normal quartic scroll $R$. We have

$$\langle \rho, \rho \rangle = 10,$$

where $\rho$ is the cohomology class of $R$. Then the intersection matrix of $h^2$ and $\rho$ is

$$\begin{pmatrix} 3 & 4 \\ 4 & 10 \end{pmatrix},$$

whose determinant is 14. It can be proved that the cubics containing such a scroll form an irreducible divisor in $\mathcal{C}$, which is then $\mathcal{C}_{14}$. Moreover, for any such $X$, the family quartic scroll contained in $X$ is the associated K3 surface.

### 3.3.4 d=20: Cubic fourfolds containing a Veronese surface

Let $X$ be a smooth cubic fourfold containing a Veronese surface $V$. We have

$$\langle v, v \rangle = 12,$$

where $v$ is the cohomology class of $V$. Then the intersection matrix of $h^2$ and $v$ is

$$\begin{pmatrix} 3 & 4 \\ 4 & 12 \end{pmatrix},$$

whose determinant is 20. It can be proved that the cubics containing a Veronese surface form an irreducible divisor in $\mathcal{C}$, which is then $\mathcal{C}_{20}$.

### 3.3.5 d=26: Cubic fourfolds containing a 3-nodal septic scroll

This characterization was introduced by Farkas and Verra in [FV1].

**Definition 3.3.1.** A smooth septic scroll is a surface $R' \subset \mathbb{P}^8$ which is image the map

$$\phi_{|4l-3E|} : \mathbb{F}_1 = \mathrm{Bl}_o \mathbb{P}^2 \to \mathbb{P}^8$$

up to linear transformations of $\mathbb{P}^8$.

**Lemma 3.3.2** ([FV1, 3.1])**.** *Let $R' \subset \mathbb{P}^8$ be a smooth septic scroll. Let $a_1, a_2, a_3 \in$* $\mathrm{Sec}(R')$ *be general points and set $\Lambda := \mathrm{Span}(\{a_1, a_2, a_3\}) \in G(3, 9)$. Let $\pi : R' \to \mathbb{P}^5$ be the projection with center $\Lambda$. Then* $\mathrm{Im}(\pi)$ *has three non-normal nodes corresponding to three bisecant lines passing through $a_1, a_2, a_3$ and no further singularities.*

The lemma above justifies the following:

**Definition 3.3.3.** Let $R', \Lambda$ and $\pi$ as in the above lemma. Then $R := \mathrm{Im}(\pi)$ is called a 3-nodal septic scroll.

It can be proved that, for the class $\rho'$ of a 3-nodal septic scroll $R'$ contained in a smooth cubic fourfold $X$, one has:

$$\langle \rho', \rho' \rangle = 25.$$

So the intersection matrix of $h^2$ and $\rho'$ is

$$\begin{pmatrix} 3 & 7 \\ 7 & 25 \end{pmatrix},$$

whose determinant is 26. It can be proved that the cubic fourfolds containing a 3-nodal septic scroll form an irreducible divisor in $\mathcal{C}$, which is $\mathcal{C}_{26}$.

### 3.3.6 The family of divisors $\mathcal{C}_d$, $d = 2(n^2 + n + 1)$ and the rationality problem

The values $d = 2(n^2+n+1)$ with $n \geq 2$ are all admissible. Hasset showed in [Has1, Sec. 6] that there is an isomorphism

$$F(X) \cong S^{[2]}.$$

For each point of $S$ it is possible to define a curve in $S^{[2]}$:

$$\Delta_p = \{\xi \in S^{[2]} : \mathrm{Supp}(\xi) = p\}.$$

Then, for $d = 2(n^2+n+1)$[1], there is a family of curves in $F(X)$ parametrized by points of $S$. Such a curve in $F(X)$ naturally corresponds to a rational scroll contained in $X$. The associated K3 surface has the following interpretations for small values of $d = 2(n^2 + n + 1)$:

1. for $X \in \mathcal{C}_{14}$ it is the Hilbert scheme of quartic scrolls contained in $X$;
2. for $X \in \mathcal{C}_{26}$ it is the Hilbert scheme of 3-nodal septic scrolls contained in $X$ (see for example [FV1]);
3. for $X \in \mathcal{C}_{42}$ it is the Hilbert scheme of 8-nodal nonic scrolls contained in $X$ (see for example [FV2]).

The interest on these families of divisors relies on the fact that they are conjectured to contain all the rational examples. Kuznetsov gave in [Kuz] a categorical condition for the rationality:

**Conjecture 3.3.4** ([Kuz, 1.1])**.** A smooth cubic fourfold is rational if and only if its Kuznetsov component $\mathrm{Ku}(X)$ is derived equivalent ot a K3 surface.

[1] Actually this condition is stronger than having an associated K3.

Addington and Thomas proved in [AT] that the Kuznetsov condition is equivalent to the fact that $X \in \mathcal{C}_d$ for $d$ admissible. The conjecture has been proved for the initial values of $d$. Morin proved in [Mor1] that a cubic fourfold containing a quartic rational normal scroll is rational, which is the initial case for $d > 6$. Morin claimed that all cubic fourfolds contain a quartic rational normal scroll (and hence it is rational), Fano in [Fa] discovered the error in the Morin's argument. See more in the Introduction to chapter 4. This result has been recently extended by M. Bolognesi, F. Russo and G. Staglianò in [BRS]:

**Theorem 3.3.5.** *Every cubic fourfold in $\mathcal{C}_{14}$ is rational.*

In [RS1] and [RS2] F. Russo and G. Staglianò proved the rationality of cubic fourfolds for the first four cases of Hassett divisors.

**Theorem 3.3.6.**

- *Every smooth cubic fourfold in $\mathcal{C}_{26}$ and $\mathcal{C}_{38}$ is rational ([RS1]).*
- *Every smooth cubic fourfold in $\mathcal{C}_{42}$ is rational ([RS2]).*

# Chapter 4

# Pfaffian cubic fourfolds

## 4.1 Historical Introduction

In what follows a *Pfaffian cubic fourfold* $X \subset \mathbb{P}^5$ is, by definition, just a cubic fourfold whose equation is the Pfaffian of an order 6 skew symmetric matrix of linear forms. Let us also fix the following notation:

$$\mathbf{C} := |\mathcal{O}_{\mathbb{P}^5}(3)|$$

and

$$\mathbf{C}_d \subset \mathbf{C}$$

for the the closure of the family of stable cubic fourfolds $X$ defining a point, in the GIT-moduli space $\mathcal{C}$ of cubic fourfolds, of a Noether-Lefschetz divisor $\mathcal{C}_d$, (for a given integer $d$ such that the corresponding $\mathcal{C}_d$ is not empty).

We have already mentioned the well known property that $\mathcal{C}_d$ is an irreducible divisor. This implies that $\mathbf{C}_d$ is an irreducible divisor in $\mathbf{C}$. This is actually the closure of the union of the Aut $\mathbb{P}^5$-orbits of those stable cubic fourfolds $X$ defining a point of $\mathcal{C}_d$. Let us also recall that the closure of the family of Pfaffian cubic fourfolds is the irreducible divisor

$$\mathbf{C}_{14} \subset \mathbf{C}.$$

We have already considered $\mathbf{C}_{14}$ when considering the results of Beauville and Donagi on the Fano variety of lines of a smooth cubic fourfold. The family $\mathbf{C}_{14}$ actually admits very interesting characterizations and incarnations, so to say. The main results of our work, we are now going to show, are concentrated on $\mathbf{C}_{14}$, its related moduli space $\mathcal{F}_8$ of K3 polarized surfaces and on the intersection $\mathcal{C}_{14} \cap \mathcal{C}_6$ in $\mathcal{C}$. It is due to put the family of Pfaffian cubic fourfolds in a historical perspective, because of its relevance from the historical side as well. This is part of our program for this section.

Roughly speaking the family of Pfaffian cubic fourfolds generically coincides with the family of smooth cubic fourfolds containing a rational normal quartic scroll. Interestingly, the latter family is not open in $\mathbf{C}_{14}$, as proved by Bolognesi, Russo, Staglianò in [BRS]. Another well known condition, characterizing this family, is that $X \in \mathbf{C}_{14}$ contains a smooth quintic Del Pezzo surface. As Beauville shows, this latter condition is satisfied by all smooth elements of $\mathbf{C}_{14}$.

Notice also that a smooth quintic Del Pezzo surface $Y$ is contained in the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$ in $\mathbb{P}^5$ as a divisor whose bidegree is $(1,2)$. Actually, as is well known from the classical geometry of a smooth quintic Del Pezzo surface ([Dol2]), $Y$ is contained in exactly five embeddings of this type. In particular it follows that

$$X \cdot (\mathbb{P}^1 \times \mathbb{P}^2) = Y \cup R.$$

Here $R$ is a rational normal quartic scroll or a reducible degeneration of it.

The family of cubic fourfolds considered generically coincides with the Pfaffian family and it is important from several points of views. To begin from history, let us say that cubic fourfolds $X$ containing a rational normal quartic scroll $R$ were studied by Ugo Morin, with the purpose of proving the rationality of a general smooth cubic fourfold, [Mor1]. The existence of such an $R$ in $X$ implies indeed, as proven by Morin in [Mor1], that $X$ is birational to the Hilbert scheme $\mathrm{Hilb}_2 R$ of two points of $R$.

The reason is that $R$, as the quintic Del Pezzo surface, is an OADP-surface. OADP is the acronymous *one apparent double point* and means that a unique bisecant line to $R$ exists, passing thorugh a general point $x \in \mathbb{P}^5$. This property makes possible to construct a birational map

$$b : X \to \mathrm{Hilb}_2 R,$$

sending $x \in X$ to $l_x \in \mathrm{Hilb}_2 R$, where $l_x = L_x \cdot R$ and $L_x$ is the bisecant line to $R$ passing through $x$. However the proof of Morin that a general cubic fourfold contains a smooth rational quartic scroll is wrong. Indeed a parameter count he uses to deduce that a general cubic fourfold $X$ contains a rational nomal scroll, is virtually correct, but unfortunately not effective in the situation considered.

From today's point of view, notice also that a very general $X$ satisfies the Noether-Lefschetz property. Then any surface in $X$ is homologous to complete intersections and its degree is multiple of 3. Hence no rational normal quartic scroll $R$ exists in a general $X$.

Pfaffian cubic fourfolds reappeared, after some decades, in a very famous paper by Beauville and Donagi, [BD], we have just considered in a previous section. In it, as mentioned, the strict relation between a general Pfaffian cubic fourfold $X$ and a general K3 surface $S$ of genus 8 is discovered and it is shown that the Fano variety of lines $F(X)$ is biregular to $S^{[2]}$, the Hilbert scheme of two points of $S$. This property, though not written with the same language, is also proven by Fano via more classical and birational techniques. In [Fa], Fano shows that a general point of the Fano variety of lines of $F(X)$ corresponds to a line in $X$ which is contained in exactly two rational normal quartic scrolls of the Hilbert scheme of $R$. Since, as we better know nowadays, the Hilbert scheme $Hilb_R$ of $R$ is the previous K3 surface $S$, the property proven by Fano defines a map

$$f : F(X) \to S^{[2]}$$

sending the line $l \in F(X)$ to the unique unordered pair of points $r_1, r_2 \in S$, corresponding to the unordered pair of rational normal quartic scrolls $R_1, R_2$, of the Hilbert scheme $Hilb_R$, having the line $l$ as a bisecant line.

Notice that Fano's construction automatically implies that Morin's parameter count and argument does not effectively work. This remark is a motivation of Fano's paper ([Fa]).

Even more interestingly, the mentioned results imply, as we already observed elsewhere, that the Fano variety of a general cubic fourfold is a hyperkähler fourfold and a smooth deformation of $S^{[2]}$.

Actually, in the frame of Hassett theory [Has1], pure Hodge theoretic methods define a birational map

$$f : \mathcal{C}_{14} \to \mathcal{F}_8$$

which associates to the isomorphism class of a general Pfaffian cubic fourfold $X$ the isomorphism class of the K3 surface $S$, whose intermediate Hodge structure is naturally embedded in the intermediate Hodge structure of $X$. From a more geometrical point of view, $f$ associates to the isomorphism class of $X$ the isomorphism class of the unique K3 surface $S$ such that $S^{[2]} \cong F(X)$, via the previous geometric construction.

Among several other facts and properties, it is worth to point out that $\mathcal{C}_{14}$ was for longtime the only known divisor, not in the boundary of $\mathcal{C}$, was general point represents the GIT-class of a smooth rational cubic fourfold. As is well known, three new divisors with the same property have been very recently constructed by Russo and Staglianò, see [RS1] and [RS2].

## 4.2 The general Pfaffian cubic

### 4.2.1 Lines and alternating forms of $\mathbb{P}^5$

We begin with a 6-dimensional complex vector space $V$ and the Grassmann variety of lines of the 5-dimensional projective space $\mathbb{P}(V)$. Let

$$\mathbb{G} \subset \mathbb{P}(\bigwedge^2 V)$$

be the Plücker embedding of this Grassmannian, following [BD] and [Bolognesi-Verra], we consider the space $\mathrm{Alt}(V)$ of antisymmetric linear forms

$$\mathsf{a} : V \times V \to \mathbb{C}.$$

Then we consider the natural duality

$$\mathsf{w} : \bigwedge^2 V \times \mathrm{Alt}(V) \to \mathbb{C}.$$

This is defined by setting $\mathsf{w}(v_1 \wedge v_2, a) := \mathsf{a}(v_1, v_2)$ and extending the map by linearity. For further convenience we fix the notation

$$\mathbb{P}^{14} := \mathbb{P}(\mathrm{Alt}(V)).$$

With a slight abuse a non zero vector $\mathsf{a} \in \mathrm{Alt}(V)$ and the point defined by it in $\mathbb{P}^{14}$ will be denoted in the same way. Moreover, our constant convention is to reserve the notation $\mathbb{P}^5$ for any 5-space contained in $\mathbb{P}^{14}$.

Let us consider the natural pairing

$$\mathsf{p} : \bigwedge^2 V \times \bigwedge^4 V \to \bigwedge^6 V,$$

such that $(a, b) = a \wedge b$. This induces an identitification of projective spaces

$$\mathbb{P}^{14} = \mathbb{P}(\bigwedge^4 V).$$

**Definition 4.2.1.** *The general Pfaffian cubic is the hypersurface*

$$\mathbb{D} := \{\mathsf{a} \in \mathbb{P}^{14} \mid \mathsf{a} \textit{ is degenerate}\}.$$

It is a basic property that $\mathbb{P}^{14}$ is a quasi homogeneous space with respect to the action of the projective linear group $\mathrm{Aut}\, \mathbb{P}(V)$. The set of its orbits is indeed finite and each orbit is the set of those elements a having the same rank. This is of course true for a vector space $V$ of any dimension. As is well known, the rank of an alternating form is even if $V$ has even dimension. Let us describe what happens in our case of dimension six. Let

$$\widehat{\mathbb{G}} : \{\mathsf{a} \in \mathbb{P}^{14} \mid \operatorname{rank} \mathsf{a} = 2\},$$

we just have $\widehat{\mathbb{G}} \subset \mathbb{D} \subset \mathbb{P}^{14}$ and the orbits are

$$\widehat{\mathbb{G}} \ , \ \mathbb{D} \setminus \widehat{\mathbb{G}} \ , \ \mathbb{P}^{14} \setminus \mathbb{D}.$$

**Definition 4.2.2.** *Let* $\mathsf{a} \in \mathbb{D}$. *We denote by* $\mathsf{K}_{\mathsf{a}}$ *the projectivization of* $\ker(\mathsf{a})$.

Notice that, in the space $\mathbb{P}^{14} = \mathbb{P}(\bigwedge^4 V)$, the locus $\widehat{\mathbb{G}}$ is precisely the locus of reducible non zero vectors $v_1 \wedge v_2 \wedge v_3 \wedge v_4$. On other words $\widehat{\mathbb{G}}$ is the Plücker embedding of the Grassmannian variety of 4-spaces of $V$. The geometric interpretation of the stratification is nice and very well known, therefore let us see more of it, [Russo-book], [Fulton-Lazarsfeld].

Though not strictly necessary, we fix from now on a basis $\mathsf{e}_1, \dots, \mathsf{e}_6$ of $V$. Let $\mathsf{e}_1^*, \dots, \mathsf{e}_6^*$ be its dual, then any $\mathsf{a} \in \operatorname{Alt}(V)$ is uniquely written as

$$\mathsf{a} = \sum a_{ij} \mathsf{e}_i^* \wedge \mathsf{e}_j^*$$

with $i \leq j$ and $a_{ij} = -a_{ji}$. These are Plücker coordinates for $\mathsf{a}$ and we can think of $\mathbb{P}^{14}$ as of the projective space of the antisymmetric matrices

$$A = (a_{ij}).$$

**Theorem 4.2.3.**

- *The Pfaffian of* $(a_{ij})$ *is the equation of* $\mathbb{D}$.
- $\operatorname{Sing}(\mathbb{D}) = \widehat{\mathbb{G}}$.
- $\mathbb{D}$ *is the variety of bisecant lines to* $\widehat{\mathbb{G}}$.

We recall that, for a smooth projective variety $V \subset \mathbb{P}^r$, the *Secant variety* $\operatorname{Sec}(V)$ is the union of the lines $L \subset \mathbb{P}^r$ such that the scheme $L \cdot V$ has length $\geq 2$ and the *Tangent variety* $\operatorname{Tan}(V)$ is the union of all the lines tangent to $V$. It is not difficult to see that one has the equality

$$\operatorname{Sec}(\widehat{\mathbb{G}}) = \operatorname{Tan}(\widehat{\mathbb{G}}) = \mathbb{D}.$$

Indeed let $L$ be a bisecant line to $\mathbb{G}$ in two distinct points, then the scheme $L \cdot \mathbb{D}$ has length $\geq 4$. Hence, by Bézout theorem, $L \subset \widehat{\mathbb{D}}$. The same is true for a tangent line to $\widehat{\mathbb{G}}$, since it is limit of bisecant lines. Hence it follows $\mathrm{Sec}(\widehat{\mathbb{G}}) \subseteq \mathbb{D}$. Finally let $\mathsf{a} \in \mathbb{D}$, then the matrix $\mathsf{a} = (a_{ij})$ has rank 2 or 4. If the rank is 2 then $\mathsf{a} \in \widehat{\mathbb{G}}$. If the rank is 4 then $(a_{ij}) = (a^1_{ij}) + (a^2_{ij})$, where the summands are antisymmetric matrices of rank 2. Hence $\mathsf{a}$ belongs to the line joining the two points $\mathsf{a}^k = (a^k_{ij})$, $(k = 1, 2)$, of $\widehat{\mathbb{G}}$.

Finally it is worth to recall that $\widehat{\mathbb{G}}$ is one of the four Severi varieties. These are the only smooth, irreducible projective varieties $V$ of dimension $d$ in $\mathbb{P}^n$ which are not contained in a hyperplane and satisfy the conditions

$$\frac{3}{2}d + 2 = n\ ,\ \mathrm{Sec}(V) \neq \mathbb{P}^n.$$

Their importance is due to Hartshorne's conjecture on linear normality and Zak's proof of it. This says that no $V$ as above exists such that $\frac{3}{2}d + 2 > n$ and $\mathrm{Sec}(V) \neq \mathbb{P}^n$.

Keeping the previous convention, let $\mathsf{p} = \sum p_{ij}\mathsf{e}_i \wedge \mathsf{e}_j$ be any vector of $\bigwedge^2 V$, so that the antisymmetric matrix of its Plücker coordinates is $(p_{ij})$. Then, in a totally analogous way, it turns out that

$$\mathbb{G} \subset \widehat{\mathbb{D}} \subset \mathbb{P}(\bigwedge^2 V).$$

Here we fix the notation $\widehat{\mathbb{D}}$ for the Pfaffian of $\det(p_{ij})$.

### 4.2.2 The birational duality

In this paragraph we recall the very special feature of the Pfaffians

$$\mathbb{D}\ \subset \mathbb{P}^{14} \text{ and } \widehat{\mathbb{D}} \subset \mathbb{P}(\bigwedge^2 V).$$

Both are indeed homaloidal hypersurfaces. For a reduced hypersurface $Y$ in $\mathbb{P}^r$, with coordinates $(z_o : \cdots : z_r)$, the partial derivatives of the equation $F$ of $Y$ define a well known rational map in the dual projective space $\mathbb{P}^{r*}$ with dual coordinates. This is of course the duality map

$$\phi :\ \mathbb{P}^r \dashrightarrow \mathbb{P}^{r*},$$

such that

$$\phi(\mathsf{z}) = \left[\frac{\partial F}{\partial z_o}(\mathsf{z}) : \cdots : \frac{\partial F}{\partial z_r}(\mathsf{z})\right].$$

Let $\mathsf{z} \in Y \setminus \mathrm{Sing}(Y)$, then $\phi(\mathsf{z})$ is defined by the coefficients of the equation

$$\sum_{i=0}^{r} \frac{\partial F}{\partial z_i}(\mathsf{z}) z_i = 0$$

of the tangent hyperplane to $V$ at z. It is clear that $\phi$ does not depend from the choice of the coordinates. Roughly speaking we can say that:

*$\phi(V)$ is the parameter space for the singular hyperplane sections of $V \setminus \mathrm{Sing}V$.*

**Definition 4.2.4.** *A reduced hypersurface $Y$ is homaloidal if $\phi$ is birational.*

Homaloidal hypersurfaces are very special and often very interesting. As is well known the restriction of $\phi$ to $Y$ is a birational morphism

$$\phi|Y : Y \to \phi(Y)$$

if $Y$ is smooth. Moreover, by Zak's theorem on tangencies, each fibre of $\phi|V$ is finite. Our situation is completely different, as we are going to see. The properties of the duality maps to be considered are fundamental to understand the geometry of the elements of the Morin hypersurface $\mathbf{C}_{14}$.

Recall that, for this section, $\mathbb{P}^{14} = \mathbb{P}(\bigwedge^2 V)$. Under the natural identifications we have, the spaces $\mathbb{P}^{14}$ and $\mathbb{P}(\bigwedge^2 V)$ are dual, with their respective Plücker coordinates $(p_{ij})$ and $(a_{ij})$. Consider

$$\mathsf{a} = (a_{ij}) \in \mathbb{P}^{14} \ , \ \mathsf{p} = (p_{ij}) \in \mathbb{P}(\bigwedge^2 V)$$

and the corresponding matrices of cofactors

$$\mathsf{a}^* = (A_{ij})^T \ , \ \mathsf{p}^* = (P_{ij})^T.$$

Then define the following rational maps

$$\partial : \mathbb{P}^{14} \dashrightarrow \mathbb{P}(\bigwedge^2 V) \ \text{ and } \ \partial^{\perp} : \mathbb{P}(\bigwedge^2 V) \dashrightarrow \mathbb{P}^{14}.$$

by setting

$$\partial(\mathsf{a}) = \mathsf{a}^* \ , \ \partial^{\perp}(\mathsf{p}) = \mathsf{p}^*.$$

It is known, and easy to check, that $\partial$ and $\partial^{\perp}$ are the duality maps of $\mathbb{D}$ and $\widehat{\mathbb{D}}$. Their description goes as follows, [Bolognesi-Verra]:

**Theorem 4.2.5.** *The rational maps $\partial$ and $\partial^{\perp}$ are the duality maps of $\mathbb{D}$ and $\widehat{\mathbb{D}}$, they are birational and each one inverse to the other one.*

Much more is available on these maps, especially from the geometrical point of view. Let us concentrate on $\partial : \mathbb{P}^{14} \to \mathbb{P}(\bigwedge^2 V)$, keeping in account that $\partial^\perp$ admits the same (dual) description [Bolognesi-Verra]:

- $\partial$ is defined by the order 2 minors of the coordinate matrix $(a_{ij})$. Hence by the equations of the Grassmannian $\widehat{\mathbb{G}}$, which is the locus of rank 2 alternating forms a. This implies that $\partial$ blows up $\widehat{\mathbb{G}}$.
- Let us consider at first the blowing up of $\mathbb{P}^{14}$ along $\widehat{\mathbb{G}}$, say
$$\sigma : \widetilde{\mathbb{P}}^{14} \to \mathbb{P}^{14}.$$
- The exceptional divisor of $\sigma$ is a $\mathbb{P}^5$-bundle
$$\sigma|\widetilde{\mathbb{D}}^\perp : \widetilde{\mathbb{D}}^\perp \to \widehat{\mathbb{G}}.$$
- The strict transform of $\mathbb{D}$ is a second interesting divisor, say
$$\widetilde{\mathbb{D}} \subset \widetilde{\mathbb{P}}^{14}.$$

Let us recall that $\partial$ has the following birational factorization.

**Theorem 4.2.6.** *One has the factorization $\partial = \sigma^\perp \circ \sigma$, where*
$$\sigma^\perp : \widetilde{\mathbb{P}}^{14} \to \mathbb{P}(\bigwedge^2 V)$$
*is the blowing up of $\mathbb{G}$ and $\widetilde{\mathbb{D}}$ is its exceptional divisor.*

These are the properties we had to remind in order to describe:

- The family of the tangent hyperplanes to the Pfaffian $\mathbb{D}$,
- an important family of rational normal quartic cones in $\mathbb{D}$.

**Remark 4.2.7.** A direct computational description of the birational map $\partial$, including its special fibres, is available and not difficult.

### 4.2.3 The family of tangent hyperplanes to $\mathbb{D}$

Let $\mathsf{a} = (a_{ij}) \in \mathbb{P}^{14} \setminus \mathbb{D}$, then the polar hyperplane $H_\mathsf{a}$ to $\mathbb{D}$ at $\mathsf{a}$ is uniquely defined, in the obvious Plücker coordinates $(x_{ij})$, by the equation

$$\sum A_{ij} x_{ij} = 0,$$

where $(A_{ij}) = \partial(\mathsf{a})$. Now the birational duality described in the previous paragraph just means that the point $\mathsf{a}$ is uniquely reconstructed from $H_\mathsf{a}$. In other words, since $\mathbb{D}$ is a homaloidal polynomial, the rational map

$$\mathsf{a} \in \mathbb{P}^{14} \longrightarrow H_\mathsf{a} \in \mathbb{P}(\bigwedge^2 V)$$

is birational. More precisely, we know from the previous paragraph that $\partial$ is biregular at any point $\mathsf{a} \in \mathbb{P}^{14} \setminus \mathbb{D}$, which implies that $\mathsf{a}$ is the unique point having the hyperplane $H_\mathsf{a}$ as the associated polar hyperplane.

Let us also recall that $\mathbb{P}^{14} \setminus \mathbb{D}$ is an orbit of the action of the projective linear group on $\mathbb{P}^{14}$. Then it clearly follows that all the hyperplane sections $H_\mathsf{a} \cap \mathbb{D}$ are projectively equivalent.

In particular these just form the set of *transversal hyperplanes sections* of $\mathbb{D}$. Our interest is however to *tangent hyperplane sections* to $\mathbb{D} \setminus \mathrm{Sing}(\mathbb{D})$. To understand these hyperplane sections

$$H_\mathsf{a} \cap \mathbb{D}, \;\; \mathsf{a} \in \mathbb{D} \setminus \mathrm{Sing}\, \mathbb{D},$$

we have to recall the structure of the rational map $\partial|\mathbb{D}$. After the due description of it, the outcome about $\partial|\mathbb{D}$ can be summarized as follows.

**Theorem 4.2.8.** *Each point $\mathsf{p} \in \mathbb{G}$ defines a unique hyperplane $H_\mathsf{p}$ which is tangent to $\mathbb{D}$ along the linear span of the fibre at $\mathsf{p}$ of*

$$\partial|\mathbb{D} \to \mathbb{G}.$$

Let us summarize more precisely the steps to reach the previous theorem. Let us also say that the proof of it is a simple computation, in the Plücker coordinates we have fixed. In particular notice that the family of tangent hyperplanes we are considering is one of the three orbits of the action of $\mathrm{Aut}\, \mathbb{P}(V)$ on $|\mathcal{O}_{\mathbb{P}^{14}}(1)|$. The hyperplanes of this family are obviously parametrized, not birationally, by the points of $\mathbb{D} \setminus \widehat{\mathbb{G}}$. Since the latter set is an orbit, we can conveniently choose the coordinates for $\mathsf{a}$. On the other hand, as we are going now to see, the tangent hyperplane at $\mathsf{a}$ is uniquely reconstructed from the point $\mathsf{p} = \partial(\mathsf{a}) \in \mathbb{G}$.

Let $\mathsf{p} = v_1 \wedge v_2$ then, in the exterior algebra $\bigwedge^* V = \oplus_{\mathbb{Z}} \bigwedge^i V$ we can consider the ideal $\mathsf{I}$ generated by $v_1, v_2$, that is the orthogonal space of $\{v_1, v_2\}$ with respect to the wedge product. The degree 4 summand $\mathsf{I}(4)$ of $I_{v_1,v_2}$ has the canonical decomposition

$$\mathsf{I}(4) = (\bigwedge^2 V \wedge \mathsf{p}) \bigoplus (\bigwedge^3 V \wedge V_{\mathsf{p}})/(\bigwedge^2 V \wedge \mathsf{p}),$$

where $V_{\mathsf{p}} \subset$ is the 2-dimensional space whose projectivization is the line defined by $\mathsf{p}$. Notice that the above vector space has dimension 14 and it is the orthogonal space of $\mathsf{p}$.

### 4.2.4 The family of rational normal quartic cones

After the description of the family of tangent hyperplanes to $\mathbb{D} \setminus \operatorname{Sing} \mathbb{D}$, and of their peculiar geometry, it is now the moment for considering the family of rational normal quartic cones which are naturally associated to the tangent hyperplanes.

The latter family, as we will see, plays a very important role in order to characterize the family of those cubic fourfolds which are linear sections of $\mathbb{D}$, singular at some point of $\mathbb{D} \setminus \operatorname{Sing}\mathbb{D}$. The existence of these scrolls is implicitly considered in [BD]. In the preprint [Bolognesi-Verra] their construction is generalized and made very explicit. We are grateful to the authors for the information about and for the possibility of using this preprint.

Let us consider any point $\mathsf{p} \in \mathbb{G}$ and the line defined by it, say

$$\ell_{\mathsf{p}} \subset \mathbb{P}(V).$$

We have seen in the previous paragraph how $\ell_{\mathsf{p}}$ uniquely defines a tangent hyperplane $H_{\mathsf{p}}$. Actually this is characterized by the condition

$$H_{\mathsf{p}} \cap \widehat{\mathbb{G}} = \{\mathsf{a} \in \mathbb{D} \mid \mathsf{p} \wedge \mathsf{a} = 0\}.$$

In other words, under the usual duality $\bigwedge^2 V \times \bigwedge^4 V \to \bigwedge^6 V$, we have

$$H_{\mathsf{p}} = \{p\}^{\perp}.$$

Here we adopt the following, more general, definition.

**Definition 4.2.9.** Let $S \subset \mathbb{P}(\bigwedge^2 V)$ any subset, the orthogonal of $S$ is the linear space

$$S^{\perp} = \bigcap_{\mathsf{p} \in S} H_{\mathsf{p}}.$$

We will be mainly interested to the case where $S$ is a linear section of $\mathbb{G}$ and especially a $K3$ surface. Let $\mathsf{p} \in \mathbb{G}$, then the geometric description of $\mathsf{H}_\mathsf{p} \cap \mathbb{D}$ goes as follows. Consider the Schubert hyperplane section

$$Q_\mathsf{p} := \{\mathsf{a} \in \widehat{\mathbb{G}} \mid \ell_\mathsf{p} \cap \mathsf{K}_\mathsf{a} \neq \emptyset\},$$

that is the family of alternating forms whose projectivized kernel $\mathsf{K}_\mathsf{a}$ has non empty intersection with $\ell_\mathsf{p}$. Notice that $Q_\mathsf{p}$ contains

$$Q^4_\mathsf{p} := Q_\mathsf{p} \cap \widehat{\mathbb{G}} = \{\mathsf{a} \in \widehat{\mathbb{G}} \mid \ell_\mathsf{p} \subset \mathsf{K}_\mathsf{a}\}$$

This is just the family of alternating forms $\mathsf{a}$ of rank 2 whose projectivized kernel $\mathsf{K}_\mathsf{a}$ contains the line $\ell_\mathsf{p}$. An exercise in linear algebra shows that $Q^4_\mathsf{p}$ is a smooth quadric in a 5-space contained in $\mathbb{D}$. More precisely let $V_\mathsf{p} \subset V$ be the space such that $\mathbb{P}(V_\mathsf{p}) = \ell_\mathsf{p}$, then we have a linear map

$$- \wedge \mathsf{p} : \bigwedge^2 V \to \bigwedge^4 V$$

defined by the wedge product with $\mathsf{p}$. This linear map induces a canonical isomorphism $\bigwedge^2(V/V_\mathsf{p}) \to \bigwedge^2 V \wedge \mathsf{p}$ and a linear embedding

$$e_\mathsf{p} : \mathbb{P}(\bigwedge^2(V/V_\mathsf{p})) \to \mathbb{P}(\bigwedge^4 V) = \mathbb{P}^{14}.$$

**Lemma 4.2.10.** *The linear span $\langle Q^4_\mathsf{p}\rangle$ of $Q^4_\mathsf{p}$ is the image of $e_\mathsf{p}$, moreover*

$$Q^4_\mathsf{p} = \langle Q^4_\mathsf{p}\rangle \cap \widehat{\mathbb{G}}$$

*Proof.* Up to a base change we can assume that the vector $e_5 \wedge e_6$ defines the point $\mathsf{p}$. Then a basis of the space $\mathrm{Im}(- \wedge \mathsf{p})$ is $\{e_i \wedge e_j \wedge \mathsf{p} : 1 \leq i < j \leq 4\}$. This easily implies the statement. $\square$

Let us fix our notation as follows:

**Definition 4.2.11.** $\mathbb{P}^5_\mathsf{p} := \langle Q^4_\mathsf{p}\rangle$.

So $\mathbb{P}^5_\mathsf{p}$ is the linear space *contained in* $\mathbb{D}$ and cutting $Q^4_\mathsf{p}$ on $\widehat{\mathbb{G}}$. Let us continue to geometrically describe $Q_\mathsf{p}$ though, as we will see, the computational construction of it as a determinantal quartic cone is easy. Let

$$\mathsf{a} \in Q_\mathsf{p} \setminus Q^4_\mathsf{p},$$

then $\mathsf{a}$ is the parameter point of a line $\mathsf{K}_\mathsf{a} \subset \mathbb{P}(V)$ such that

$$\mathsf{K}_\mathsf{a} \cap \ell_\mathsf{p} := \{x_\mathsf{a}\}.$$

Indeed, by definition, $\mathsf{a} \in Q_{\mathsf{p}} \setminus Q^4_{\mathsf{p}}$ iff $\mathsf{a}$ has rank 4 and the line $\mathsf{K_a}$ meets the line $\ell_{\mathsf{p}}$ exactly in one point. Let

$$y_{\mathsf{a}} \subset \mathbb{P}(V/V_{\mathsf{p}})$$

be the projection of $\mathsf{K_a}$ from $\mathbb{P}^1 := \ell_{\mathsf{p}}$ into the 3-space 'at infinity'

$$\mathbb{P}^3 := \mathbb{P}(V/V_{\mathsf{p}}).$$

Then, starting from $\mathsf{a}$, we have constructed the point

$$(x_{\mathsf{a}}, y_{\mathsf{a}}) \in \ell_{\mathsf{p}} \times \mathbb{P}(V/V_{\mathsf{p}}) := \mathbb{P}^1 \times \mathbb{P}^3$$

which is uniquely defined from $\mathsf{p}$. Let us consider the projection of center

$$\mathbb{P}^5_{\mathsf{p}} = \mathbb{P}(\bigwedge^2 V \wedge \mathsf{p}) \subset H_{\mathsf{p}} = \mathbb{P}(\{\mathsf{p}\}^{\perp}).$$

The image of this projection is the space 'at infinity'

$$\mathbb{P}^7_{\mathsf{p}} := \mathbb{P}(\{\mathsf{p}\}^{\perp}/(\bigwedge^2 V \wedge \mathsf{p})).$$

We denote this projection as:

$$\lambda_{\mathsf{p}} : H_{\mathsf{p}} \to \mathbb{P}^7.$$

**Theorem 4.2.12.**

- $\lambda_{\mathsf{p}}(Q_{\mathsf{p}}) = \mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}^7$.
- $Q_{\mathsf{p}}$ *is a quartic cone over it.*
- $H_{\mathsf{p}}$ *is characterized by* $Q_{\mathsf{a}}$.

The theorem implies that $Q_{\mathsf{p}}$ is a quartic cone of vertex the 5-space containing $Q^4_{\mathsf{p}}$ over the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^3$. Since $\mathbb{D} \setminus \widehat{\mathbb{G}}$ is a orbit of the action of Aut$\mathbb{P}(V)$ on $\mathbb{H}_{\mathsf{a}}$, the construction of this cone is projectively invariant. In the previous Plücker coordinates let $\mathsf{a}$ be the antisymmetric matrix $(a_{ij})$ such that $a_{12} = 0$. This is the hyperplane $H_{\mathsf{p}}$ defined by the line $\ell_{\mathsf{p}}$ with $\mathsf{p} = e_1 \wedge e_2$ ........ We can write explicitly the equation of this cone.

The inncidence correspondence, (cfr.[Bolognesi-Verra]):

$$\tilde{\mathbb{Q}} := \{(x, \mathsf{a}) \in \ell_{\mathsf{p}} \times \mathbb{P}^{14} \,|x \in \mathsf{K_a}\}.$$

is defined by bilinear exquations, as we will explain soon. Assume $\mathsf{p} = e_5 \wedge e_6$, then a point

$$\lambda e_5 + \mu e_6$$

of this line is a vector of $V$ of components

$$(0, 0, 0, 0, \lambda, \mu)$$

Let $A = (a_{ij})$ be the matrix of Plücker coordinates on $\mathbb{P}^{14}$:

$$A \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} c_{15}\lambda + c_{16}\mu \\ c_{25}\lambda + c_{26}\mu \\ c_{35}\lambda + c_{36}\mu \\ c_{45}\lambda + c_{46}\mu \\ \lambda c_{56} \\ -\mu c_{56} \end{pmatrix}$$

The incidence correspondence is just defined as the locus of the solutions of the previous system of linear equations. Eliminating $\lambda, \mu$ we obtain the equations of the projection of the incidence correspondence $\tilde{\mathbb{Q}}$ in $\mathbb{P}^{14}$:

$$c_{56} = 0\,,\ c_{i,5}c_{j6} - c_{i6}c_{j5} = 0\,,\ 1 \leq i < j \leq 4.$$

This is the rational normal quartic cone

$$Q_{\mathsf{p}} \subset \mathbb{P}^{14}.$$

**Theorem 4.2.13.** $\mathbb{D}$ *contains the family of* 10*-dimensional rational normal quartic cones over the Segre product* $\mathbb{P}^1 \times \mathbb{P}^3$*:*

$$\{Q_{\mathsf{p}} : \mathsf{p} \in \mathbb{G}\},$$

*parametrized by the Grassmannian* $\mathbb{G}$*. Each of these defines a hyperplane* $H_{\mathsf{p}}$*, tangent to* $\mathbb{D}$ *along its* 5*-dimensional vertex* $\mathbb{P}^5_{\mathsf{p}}$*.*

*Proof.* Eliminating $\lambda, \mu$ we have at first $c_{56} = 0$, which is the equation of the hyperplane $H_{\mathsf{p}}$ when $\mathsf{p} = e_5 \wedge e_6$. Moreover the vanishing of the other $2 \times 2$ minors of the same matrix defines the Segre product $\mathbb{P}^1 \times \mathbb{P}^3$ in the coordinates

$$(c_{15} : c_{16} : c_{25} : c_{26} : c_{35} : c_{36} : c_{45} : c_{46}).$$

Indeed this property is well known for the Segre product $\mathbb{P}^1 \times \mathbb{P}^r \subset \mathbb{P}^{2r-1}$: the order 2 minors of the analogous $(r+1) \times 2$ matrix $M = (c_{ij})$, with $i = 1 \ldots r, j = 1, 2$, are the equations of $\mathbb{P}^1 \times \mathbb{P}^r$ (see for example [Har1]). □

## 4.3 Pfaffian cubic fourfolds and K3 surfaces

### 4.3.1 The family of Pfaffian cubic fourfolds

In what follows we concentrate our attention on the family of 5-dimensional linear sections of the cubic Pfaffian, say

$$X = L \cap \mathbb{D},$$

where $L \cong \mathbb{P}^5$.

**Definition 4.3.1.** *Any $X$ as above is a Pfaffian cubic fourfold.*

Clearly the equation of $X$ is the Pfaffian of an antisymmetric matrix of linear forms

$$c_{ij} = c_{ij1}t_1 + \cdots + a_{ij6}t_6,$$

where $(t_1 : \cdots : t_6)$ are projective coordinates on $\mathbb{P}^5$. In a more abstract way we can consider the universal 5-space

$$u : \mathbb{P} \to \mathbb{G}_{pf},$$

where $\mathbb{G}_{pf}$ denotes the Grassmannian of the 5-spaces in $\mathbb{P}^{14}$.

### 4.3.2 The family of associated K3 surfaces

Let $X = L \cap \mathbb{D}$ be a general Pfaffian cubic fourfold. Then $S := L^{\perp} \cap \mathbb{G}$ is a K3 surface of degree 14.

**Theorem 4.3.2** ([BD, 5])**.** *For a generally chosen Pfaffian cubic fourfold $X = L \cap \mathbb{D}$, the surface $S = L^{\perp} \cap \mathbb{G}$ is a K3 surface verifying the relation:*

$$S^{[2]} \cong F(X).$$

The proof gives the explicit correspondence $S^{[2]} \to F(X)$.

*Proof.* For a general pair of points of $\mathbb{G} \subseteq \mathbb{P}\left(\bigwedge^2 V\right)$ $P = v_1 \wedge v_2$ and $Q = v_3 \wedge v_4$, there is a well defined subspace of $\mathbb{P}\left(\bigwedge^4 V\right)$ given by forms identically vanishing on $\mathrm{Span}(v_1, v_2, v_3, v_4)$. This is the codimension 6 linear subspace

$$L_{\{P,Q\}} = \{[\phi] : \phi \wedge v_i \wedge v_j = 0 \text{ for } 1 \leq i < j \leq 4\} \subset \mathbb{D}.$$

Suppose now that it is fixed a 5-dimension linear subspace $L \subset \mathbb{P}\left(\bigwedge^4 V\right)$ and $P, Q \in S = L^\perp \cap \mathbb{G}$. Then $L$ and $L_{P,Q}$ are both contained in the 12-dimensional projective space $\{P, Q\}^\perp$. So, for a general choice of $P$ and $Q$ $L \cap L_{P,Q}$ is a straight line contained in $X$. Conversely, given a straight line $\ell \subset X$, there is a unique 4-dimensional space $W \subset V$ such that $W$ is isotropic for all the forms in $\ell$, i.e.

$$\phi \wedge w_1 \wedge w_2 = 0 \text{ for all } \phi \in \ell.$$

Let $K \subset \mathbb{P}(\bigwedge^2 V)$ be the 5-space containing $G(2, W)$. Then $L^\perp \cap K$ is a straight line cutting the quadric $G(2, W)$ in two points $P, Q$ such that $\mathrm{Span}(P, Q) = W$. $\square$

## 4.4 Geometry of Pfaffian cubic fourfolds

### 4.4.1 Quartic scrolls

Let $L \subset \mathbb{P}(\bigwedge^4 V)$ be a 5-space. Denote

$$X := L \cap \mathbb{D}, S := L^\perp \cap \mathbb{G}.$$

Recall from theorem 4.2.13 that for any point $\mathsf{p} \in S \subset \mathbb{G}$ there is an associated 10-dimensional cone $Q_\mathsf{p}$ of vertex $\mathbb{P}^5_\mathsf{p}$ over the Segre variety $\mathbb{P}^1 \times \mathbb{P}^3$.

For a general choice of $L$, $L$ is disjoint from $\mathbb{P}^5_\mathsf{p}$ for any p $\in$ S. Note that $\langle L Q_\mathsf{p} \rangle = \{\mathsf{p}\}^\perp$. It follows that $L \cap Q_\mathsf{p}$ is isomorphic to a double hyperplane section of $\mathbb{P}^1 \times \mathbb{P}^3$. For a general choice of $L$ it is a smooth minimal surface of degree 4, hence it is a rational normal scroll.

The construction given above gives a one-to-one correspondence between points of $S$ and the Hilbert scheme of quartic scrolls contained in $X$. Recall that there is an isomorphism

$$\mathbb{G} \to \{\text{Cones of vertex } \mathbb{P}^5 \text{ over } \mathbb{P}^1 \times \mathbb{P}^3 \text{ contained in } \mathbb{D}\},$$

which maps $\mathsf{p}$ to $Q_\mathsf{p}$. The inverse map associates a cone $Q$ to the point $\mathsf{p}_Q \in \mathbb{G}$ determined by the property $T_{\mathsf{p}_Q}\mathbb{G} = \mathrm{Sing}(Q)^\perp$. The restriction of this map to $S$ composed with the intersection with $L$ gives the desired correspondence.

### 4.4.2 Quintic scrolls and quintic Del Pezzo surfaces

Let $X$ be a cubic fourfold and $R \subset X$ a general smooth quartic scroll contained in it. Then $R$ arises as a divisor of a cubic Segre variety $T \cong \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ in two different ways:

(i) As a divisor of type $(0,2)$. In this case the residual intersecion of $T \cap X$ is a quintic scroll;

(ii) As a divisor of type $(2,1)$. In this case the residual intersection of $T \cap X$ is a quintic Del Pezzo surface.

**Remark 4.4.1.** For any general quartic scroll $R$, there is only a cubic Segre variety $T$ such that $R \subset T$ is a divisor of type $(0,2)$: in fact, taken the pencil of conics contained in $R$, $T$ is the union of the planes generated by those conics.

It follows that there is a one-to-one correspondence between quartic scrolls contained in $X$ and quintic scrolls contained in $X$.

# Chapter 5

# Pfaffian and nodal cubic fourfolds

## 5.1 Introduction and preliminaries

In this part of the thesis we concentrate our attention on two divisors in the GIT-compactification $\mathcal{C}$ of the moduli space of cubic fourfolds with at worst simple singularities) , constructed by Laza in [Laz1]. The divisors to be considered are the Morin divisor $\mathcal{C}_{14}$ and the boundary divisor $\mathcal{C}_6$, whose general element is the isomorphism class of a cubic fourfold $X$ such that Sing$X$ is one ordinary double point. As already mentioned these two divisors are irreducible. We will prove that

$$\mathcal{C}_6 \cap \mathcal{C}_{14} = \mathcal{A} \cup \mathcal{B} \cup \mathcal{N},$$

where $\mathcal{A}$, $\mathcal{B}$ are irreducible, 18-dimensional components of $\mathcal{C}_6 \cap \mathcal{C}_{14}$ and a general point $x$ of $\mathcal{A} \cup \mathcal{B}$ is defined by a 1-nodal and Pfaffian cubic fourfold. Moreover $\mathcal{N}$ is possibly empty or contained $\mathcal{C}_{14} - \mathcal{C}_{14}^{Pf}$, where

$$\mathcal{C}_{14}^{Pf} \subset \mathcal{C}_{14}$$

is the constructible set of points defined by Pfaffian fourfolds. Then its complement is, so to say, the boundary locus made up by points defined by cubic fourfolds whose equation is not Pfaffian, though limit of Pfaffian equations.

We do not address in this thesis the proof of the plausible equality

$$\mathcal{C}_6 \cap \mathcal{C}_{14} = \mathcal{A} \cup \mathcal{B}.$$

This somehow relates to a different and possible investigation, concerning the structure of the boundary locus $\mathcal{C}_{14}^{Pf} \subset \mathcal{C}_{14}$. We are grateful to professor Michele Bolognesi for his advice on the subject and useful comments.

The splitting of $\mathcal{C}_6 \cap \mathcal{C}_{14}$ is a consequence of the following facts. Let

$$X = \mathbb{D} \cap \mathbb{P}^5$$

be a general singular linear section of dimension 4, then $X$ has one ordinary node $o$ as its unique singular point and it is obviously clear that

- *either* $\mathsf{o} \in \operatorname{Sing} \mathbb{D}$
- *or* $\mathsf{o} \in \mathbb{D} \setminus \operatorname{Sing} \mathbb{D}$.

Counting dimensions, these conditions define two locally closed sets which are expected to be of codimension one in $\mathcal{C}_{14}$, hence two in $\mathcal{C}$. Passing to their closures one obtains the mentioned irreducible components $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{C}_6 \cap \mathcal{C}_{14}$. Let us fix further our notation. We will consider the Grassmannian $\mathbb{G}_{Pf}$ of 5-dimensional linear subspaces of $\mathbb{P}^{14}$. For each $\mathsf{z} \in \mathbb{G}_{Pf}$ we denote by $\mathbb{P}^5_{\mathsf{z}}$ the 5-space in $\mathbb{P}^{14}$ whose parameter point is z.

**Definition 5.1.1.** *The universal* 5*-space over* $\mathbb{G}_{Pf}$ *is the correspondence*

$$\mathbb{P} = \{(x, z) \in \mathbb{P}^{14} \times \mathbb{G}_{Pf} \mid x \in \mathbb{P}^5_z\}$$

*endowed with its natural projection morphism* $u : \mathbb{P} \to \mathbb{G}_{Pf}$.

Of course $u : \mathbb{P} \to \mathbb{G}_{Pf}$ is a $\mathbb{P}^5$-bundle with fibre $\mathbb{P}^5_z$ at $z$. More precisely it is the projective bundle associated to the universal bundle $\mathcal{U}$ of $\mathbb{G}_{Pf}$. From now on we will keep the notation $\mathsf{U}$ for the open subset

$$\mathsf{U} \subset \mathbb{G}_{Pf},$$

parametrizing the 5-spaces which are not in $\mathbb{D}$. Over $\mathsf{U}$ we define the flat family of Pfaffian cubic fourfolds

$$\mathcal{X} = \{(x, \mathsf{z}) \in \mathbb{D} \times \mathsf{U} \mid x \in \mathbb{P}^5_z \cap \mathbb{D}\},$$

together with its two natural projections

$$\pi_1 : \mathcal{X} \to \mathbb{D},\ \pi_2 : \mathcal{X} \to \mathbb{P}^5_{\mathsf{z}}.$$

Let $\mathsf{z} \in \mathsf{U}$ then $\mathcal{X}_{\mathsf{z}}$ will denote the fibre of $v$ at z. It is clear that the family

$$v : \mathcal{X} \to \mathsf{U}$$

contains all the Pfaffian cubic fourfolds up to projective equivalence. In particular $\mathsf{U}$ dominates the Morin divisor $\mathcal{C}_{14}$ via the moduli map.

**Definition 5.1.2.**

- $\mathsf{A}$ *is the closure in* $\mathbb{G}_{Pf}$ *of:*

$$\{z \in \mathsf{U} \mid \mathcal{X}_z \cap \operatorname{Sing} \mathbb{D} = \emptyset \textit{ and } \operatorname{Sing} \mathcal{X}_\mathsf{z} \neq \emptyset\}.$$

- $\mathsf{B}$ *is the closure in* $\mathbb{G}_{Pf}$ *of:*

$$\{z \in \mathsf{U} \mid \mathcal{X}_\mathsf{z} \cap \operatorname{Sing} \mathbb{D} \neq \emptyset\}.$$

Obviously $\mathcal{X}_\mathsf{z}$ is singular in both the cases.

**Lemma 5.1.3.** $\mathsf{A}$ *and* $\mathsf{B}$ *are irreducible divisors in the Grassmannian* $\mathbb{G}_{Pf}$.

*Proof.* $\mathsf{B}$ is a Grassmannian bundle over $\operatorname{Sing}(\mathbb{D})$ with fibre isomorphic to $G(5,14) \cong \mathbb{G}(4,13)$. $\operatorname{Sing} \mathbb{D}$ is biregular to the Grassmannian $\mathbb{G}$, hence it is irreducible. Therefore $\mathsf{B}$ is irreducible too. To show that $\mathsf{A}$ is irreducible consider the correspondence

$$\mathcal{X}^s := \{(x, \mathsf{z}) \in \mathcal{X} \mid x \in (\operatorname{Sing} \mathcal{X}_\mathsf{z}) \cap (\mathbb{D} \setminus \operatorname{Sing} \mathbb{D})\}$$

and the projection map $\phi : \mathcal{X}^s \to \mathbb{D} \setminus \operatorname{Sing} \mathbb{D}$. This is surjective, moreover its fibre at $x$ is biregular to the Grassmannian $G(5,14)$. Indeed $\mathbb{P}^5 \cap \mathbb{D}$ is singular at $x \in \mathbb{D} \setminus \operatorname{Sing} \mathbb{D}$ iff $x \in \mathbb{P}^5 \subset \mathsf{T}_{\mathbb{D},x}$, the latter being the tangent hyperplane to $\mathbb{D}$ at $x$. In particular each fibre of the projection

$$\phi : \mathcal{X}^s \to \mathbb{D} \setminus \operatorname{Sing} \mathbb{D}$$

is irreducible, so that $\mathcal{X}^s$ and $\phi(\mathcal{X}^s)$ are irreducible. Now just observe that

$$\phi(\mathcal{X}^s) = \{z \in \mathsf{U} \mid \operatorname{Sing} \mathcal{X}_z \cap \mathbb{D} \setminus \operatorname{Sing} \mathbb{D} \neq \emptyset\}.$$

Hence its closure is $\mathsf{A}$, which is therefore irreducible. □

**Lemma 5.1.4.** *Let* $z \in \mathsf{A} \cup \mathsf{B}$ *be general, then* $\mathcal{X}_z$ *is a* 1*-nodal cubic fourfold.*

*Proof.* Let $x \in \mathbb{D}$, then the family of the 5-spaces $\mathbb{P}^5$, passing through $x$ and contained in the tangent projective space $T_x\mathbb{D}$, has $x$ as its only common point. Applying Bertini theorem to a general $\mathbb{P}^5$ in the family, the section $X = \mathbb{P}^5 \cap \mathbb{D}$ is smooth along $X \setminus \{x\}$. Now let $\mathsf{Q}_x$ be, the quadratic tangent cone to $\mathbb{D}$ at $x$. Then its rank is 6 if $x \in \operatorname{Sing} \mathbb{D}$, cfr. [Mu1] Prop. 1.4. If $x \notin \operatorname{Sing} \mathbb{D}$ then $\mathsf{Q}_x$ is a quadric of rank 8 in the hyperplane $\mathsf{T}_x\mathbb{D}$. In both cases it follows that $\mathbb{P}^5 \cap \mathsf{Q}_x$ has rank 5. This implies the statement. □

**Definition 5.1.5.** *$\mathcal{A}$ and $\mathcal{B}$ are the images in $\mathcal{C}$, via the natural moduli map, of the hypersurfaces* $\mathsf{A}$ *and* $\mathsf{B}$ *of the Grassmannian* $\mathbf{G}_{Pf}$.

Now, before passing to next section, it is useful to recall some of the consequences of Beauville-Donagi paper [BD] and Hassett's Theory [Has2] for the situation we are discussing. Let $\mathsf{z} \in \mathbb{G}_{Pf}$ we fix the notation

$$\mathbb{P}^8_{\mathsf{z}}$$

for the 8-space of $\mathbb{P}(\bigwedge^2 V)$ which is orthogonal to $\mathbb{P}^5_{\mathsf{z}}$, with respect to our pairing $w : \bigwedge^2 V \times \bigwedge^4 V \to \mathbb{C}$. Let us consider $\mathrm{Aut}(\mathbb{G}) = \mathrm{Aut}(\mathbb{P}(V))$ and

$$\mathsf{U}' = \{\mathsf{z} \in \mathbb{G}_{Pf} \mid \mathbb{P}^8_{\mathsf{z}} \cdot \mathbb{G} \text{ is proper and } \mathrm{Aut}(\mathbb{P}(V)) - \text{stable}\},$$

then the GIT-quotient $U' /\!/ \mathrm{Aut}\,\mathbb{P}(V)$ is a projective compactification of the moduli space of $K3$ surfaces polarized in genus 8, cfr. [Laz3]. With some abuse, we will still denote it as $\mathcal{F}_8$. Consider in $\mathbb{G} \times \mathsf{U}'$ the correspondence

$$\mathcal{S} = \{(p, \mathsf{z}) \in \mathbb{G} \times \mathsf{U}' \mid p \in \mathbb{G} \cdot \mathbb{P}^8_{\mathsf{z}}\},$$

and the integral, flat family induced by the projection map

$$v' : \mathcal{S}' \to \mathsf{U}'.$$

Then its moduli map $m' : \mathsf{U}' \to \mathcal{F}_8$ is dominant with fibre $\mathrm{Aut}\,\mathbb{P}(V)$. Let $\mathcal{S}_{\mathsf{z}} = v'^*(\mathsf{z})$, then $m'(\mathsf{z})$ is the moduli point of the pair $(\mathcal{S}_{\mathsf{z}}, \mathcal{O}_{\mathcal{S}_{\mathsf{z}}}(1))$. We already mentioned that $m : \mathsf{U} \to \mathcal{C}_{14}$ is dominant with fibre $\mathrm{Aut}\,\mathbb{P}(V)$, in a completely analogous way. To conclude we point out that the previous constructions and remarks define a rational map

$$f : \mathcal{F}_8 \to \mathcal{C}_{14}$$

sending the moduli point of $(\mathcal{S}_{\mathsf{z}}, \mathcal{O}_{\mathcal{S}_{\mathsf{z}}}(1))$ to the moduli point of $\mathcal{X}_{\mathsf{z}}$. This is a rational map of varieties of the same dimension and it is obviously invertible. Hence it follows that $f$ is birational, cfr. [Has2].

## 5.2 The intersection $\mathcal{C}_{14} \cap \mathcal{C}_6$

Let $[X] \in \mathcal{A} \cup \mathcal{B}$ be a general point then $X$ is 1-nodal and the fibre at $[X]$ of the moduli map $m_{|\mathsf{A}\cup\mathsf{B}} : \mathsf{A} \cup \mathsf{B} \to \mathcal{C}$ is $\mathrm{Aut}\,\mathbb{P}(V)$, which is isomorphic to the projective linear group $\mathrm{PGL}(6)$. Counting dimensions it follows

$$\dim \mathcal{A} = \dim \mathcal{B} = 53 - 35 = 18.$$

As a consequence of the latter equality we can deduce that $\mathcal{A}$ and $\mathcal{B}$ are irreducible components of $\mathcal{C}_{14} \cap \mathcal{C}_6$ and hence that

$$\mathcal{A} \cup \mathcal{B} \subseteq \mathcal{C}_{14} \cap \mathcal{C}_6 \subset \mathcal{C}.$$

As announced, we do not address the quite natural *equality statement*, that is $\mathcal{A} \cup \mathcal{B} = \mathcal{C}_{14} \cap \mathcal{C}_6$, but a weaker version of it.

**Theorem 5.2.1.** $\mathcal{A}^{Pf} \cup \mathcal{B}^{Pf} = \mathcal{C}_{14}^{Pf} \cap \mathcal{C}_6$.

**Remark 5.2.2.** Since $\mathcal{C}_{14}^{Pf}$ is not open in $\mathcal{C}_{14}$, the equality $\mathcal{A} \cup \mathcal{B} =$

## 5.3 Theorems A and B: the program

Now our purpose is to give appropriate characterizations of the divisors $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{C}_{14}$, deducing in particular that they are distinct. Therefore let us consider a general Pfaffian and 1-nodal cubic fourfold

$$X = \mathbb{D} \cap \mathbb{P}^5$$

and let us fix coordinates $(t_1 : \cdots : t_6)$ on $\mathbb{P}^5$ so that the *origin*

$$o = (0:0:0:0:0:1)$$

is the unique singular point of $X$. Then we can assume that

$$t_6 F_2 + F_3 = 0,$$

is the equation of $X$, where $F_2, F_3 \in \mathbb{C}[t_1 \ldots t_5]$ are forms respectively of degree 2 and 3. Since $F_2$ defines the tangent cone to $X$ at the node $o$, its rank is 5. Notice also that the forms $F_2$ and $F_3$ define in $\mathbb{P}^5$ the cone

$$F(X)_o := \{F_2 = F_3 = 0\},$$

which is union of the lines in $X$ passing through $o$. Let $\mathbb{P}^4_o$ be the 4-space parametrizing all lines through $o$, we fix on it coordinates

$$(t_1 : \cdots : t_5)$$

in the obvious way. This means that $(t_1 : \cdots : t_5)$ defines the line joining $o$ to $t = (t_1 : \cdots : t_5 : 0)$. Moreover we consider in $\mathbb{P}^4_o$ the intersection

$$S_o := \{F_2 = F_3 = 0\}.$$

For a general $X$ as above, $S_o$ is a smooth complete intersection that is $S_o$ is a smooth K3 surface of degree 6. Actually a general K3 surface with a degree 6 polarization is biregular to some $S_o$ as above. We fix the notation

$$H_o \in |\mathcal{O}_{S_o}(1)|$$

for its hyperplane sections. Finally we also consider, as usual, the section

$$S := X^{\perp} \cap \mathbb{G} = \mathbb{P}^{5\perp}.$$

Theorems **A** and **B**, we are going to state, summarize our program for this chapter. We want to describe $\mathcal{A}$ and $\mathcal{B}$ in many ways. To this purpose, denoting $\mathcal{A}$ or $\mathcal{B}$ by $\mathcal{D}$, we will construct a number of *locally closed* sets

$$\mathcal{V} \subset \mathcal{C}$$

of geometrical interest for $\mathcal{D}$. $\mathcal{V}$ will be irreducible of the same dimension 18 and constructed so that $\mathcal{D} \cap \mathcal{V}$ contains a non empty open set of $\mathcal{D}$. Since $\mathcal{D}$ is closed in $\mathcal{C}$, this implies $\mathcal{V} \subset \mathcal{D}$ and that a general point of $\mathcal{D}$ satisfies the geometric property defining $\mathcal{V}$. In other words:

*$\mathcal{V}$ generically coincides with $\mathcal{D}$ and the properties characterizing it will provide different geometric pictures of $\mathcal{A}$ or $\mathcal{B}$, as follows.*

## 5.4 Stating theorem A

We begin with $\mathcal{A}$. In this case the key words for defining $\mathcal{V}$ are:

*General nodal K3 surface of genus* 8, *rational normal quartic curve, cone in* $\mathbb{P}^5$ *over a rational normal quartic curve. Observe that it is unique up to* $\mathrm{PGL}(6)$ *action.*

We fix the notation $V_o$ for a non degenerate cone in $\mathbb{P}^5$, of vertex a point $o$, over a rational normal quartic curve. Then let us fix our definitions.

**Definition 5.4.1.**

$$\mathcal{V}^a_o := \{[X] \in \mathcal{C} \mid X \textit{ is a general cubic fourfold containing } V_o\}.$$

Notice that the projective tangent space at $o$ to the cone $V_o$ is $\mathbb{P}^5$. This implies that any $X$ containing $V_o$ is singular at $o$. On the other hand, it is very easy to produce a 1-nodal cubic fourfold $X$ containing $V_o$. Therefore $\mathcal{V}^a_o$ is not empty. The next lemma confirms the required property of $\mathcal{V}^a_o$.

**Lemma 5.4.2.** *$\mathcal{V}^a_o$ is irreducible of dimension* 18.

*Proof.* Notice that $V_o$ is unique up to projective equivalence in $\mathbb{P}^5$. Therefore we can fix it and then consider its ideal sheaf $\mathcal{I}$ and the linear system $|\mathcal{I}(3)|$. In particular this implies that the image of the rational moduli map

$$m : |\mathcal{I}(3)| \to \mathcal{C}$$

contains $\mathcal{V}^a_o$. Moreover, by the previous remarks, a general $X \in |\mathcal{I}(3)|$ is 1-nodal and hence $m(X) \in \mathcal{V}^a_o$. This implies that $\mathcal{V}^a_o$ and the image of $m$ have the same closure, so that $\mathcal{V}^a_o$ is irreducible. Now $V_o$ is the tautological model of $\mathbb{P}(\mathcal{E})$, where $\mathcal{E}$ is the rank two vector bundle $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(4)$. Then it follows that Aut $V_o$ has dimension $9 = h^0(\mathcal{E} \otimes \mathcal{E}^*) - 1 + \dim \text{Aut } (\mathbb{P}^1)$. Moreover $V_o$ is a cone over a smooth hyperplane section $R = \mathbb{P}^4 \cap V_o$ and this is a projectively normal curve. Hence $V_o$ is projectively normal. Let $\mathcal{I}_{R|\mathbb{P}^4}$ be its ideal sheaf in $\mathbb{P}^4$. Then one can compute that

$$h^0(\mathcal{I}(3)) = h^0(\mathcal{I}_{R|\mathbb{P}^4}(3)) + h^0(\mathcal{I}_{R|\mathbb{P}^4}(2)) = 28.$$

This implies that $\dim \mathcal{V}^a_o = \dim |\mathcal{I}(3)| - \dim \text{Aut} V_o = 27 - 9 = 18.$ $\square$

**Definition 5.4.3.**

$$\mathcal{V}^a_1 := \{[X] \in \mathcal{C}_{14} \mid \ S = L^\perp \cap \mathbb{G} \textit{ is a general, nodal linear section of } \mathbb{G} \ \}.$$

Let us make precise our generality assumption on $S$. We assume that $\text{Sing}\, S = \{p\}$, where $p$ is a node of $S$. Let $\sigma : S' \to S$ be the minimal desingularization of $S$, consider $H' \in |\sigma^* \mathcal{O}_S(1)|$ and $R' = \sigma^{-1}(p)$. Then $R'$ is a $-2$ curve biregular to $\mathbb{P}^1$, $\mathcal{O}_{S'}(H')$ is a genus 8 polarization on $S'$ and we have $H'^2 = 14$, $R'^2 = -2$, $H'R' = 0$. We will also assume that

$$\text{Pic}\, S' = \mathbb{Z}[H'] \oplus \mathbb{Z}[R'].$$

Now let $\mathbb{L}$ be an abstract lattice isometric to $\text{Pic}\, S'$ and let $\mathcal{F}_8$ be the moduli space of genus 8 K3 surfaces. Then $\mathcal{F}_8$ contains an irreducible divisor $\mathcal{F}^n_8$, whose elements are the moduli points of $K3$ surfaces whose Picard group contains a primitive embedding of $\mathbb{L}$. It follows from Mukai theory for K3 surfaces in low genus that $\mathcal{F}^n_8$ is birational to the GIT moduli space $\mathcal{M}$ of 1-nodal linear sections $S$ of $\mathbb{G}$ such that $\text{Pic}\, S' = \mathbb{Z}[H'] \oplus \mathbb{Z}[R']$.

**Definition 5.4.4.**

$$\mathcal{V}^a_1 := \{[X] \in \overline{\mathcal{C}} \mid X \in \mathsf{U} \textit{ and } [S] \in \mathcal{M}\}.$$

Since the GIT-quotient $\mathcal{M}$ is birational to $\mathcal{F}^n_8$ the next lemma follws.

**Lemma 5.4.5.** *$\mathcal{V}_1^a$ is irreducible of dimension* 18.

On the other hand let us go back to the irreducible divisor $\mathcal{C}_6$. In it there exists a non empty open set whose points $[X]$ are defined by 1-nodal cubic fourfolds $X$. Keeping the notation $o$ for the node of $X$, the equation of $X$ is

$$t_6F_2 + F_3 = 0.$$

As we know there exists a natural birational map between $\mathcal{C}_6$ and $\mathcal{F}_6$, the moduli space of smooth K3 surfaces which are complete intersection of a quadric and a cubic in $\mathbb{P}^4$. We recall that this birational map

$$\phi : \mathcal{C}_6 \to \mathcal{F}_6$$

associates to a general $[X] \in \mathcal{C}_6$ the point $[S_o] \in \mathcal{F}_6$, where $S_o$ is the smooth complete intersection $\{F_2 = F_3 = 0\}$. Notice that $[S_o]$ is a general point of $\mathcal{F}_6$. Therefore for a general 1-nodal $X$ we have

$$\text{Pic } S_o \cong \mathbb{Z}[H_o].$$

Notice that any surface $S_o$ is the image, under the projection $p_o : \mathbb{P}^5 \to \mathbb{P}^4$ of center $o$, of the cone $F(X)_o$ which is union of the lines of $X$ through $o$. In particular the surface $S_o$ contains a rational normal quartic curve

$$R \subset S_o$$

in the *special* case where $[X]$ is general in the family $\mathcal{V}_o^a$, parametrizing cubic fourfolds containing a cone $V_o$ as above.

**Definition 5.4.6.**

$\mathcal{S}_o := \{[S_o] \in \mathcal{F}_6 \mid S_o$ *is a general element containing a rational normal quartic* $R\}$.

Let $[S_o] \in \mathcal{S}_o$ be general, then $S_o$ is a smooth complete intersection and

$$\text{Pic} S_o \cong \mathbb{Z}[H_o] \oplus \mathbb{Z}[R].$$

**Definition 5.4.7.**

$$\mathcal{V}_2^a := \{[X] \in \overline{\mathcal{C}} \mid \ X \textit{ is } 1\textit{-nodal and } [S_o] \in \mathcal{S}_o\}.$$

Exactly as in the previous cases the next lemma follows.

**Lemma 5.4.8.** *$\mathcal{V}_2^a$ is irreducible of dimension* 18.

We can finally state Theorem A:

**Theorem 5.4.9** (Theorem A). *The family $\mathcal{A}$ generically coincides with*

$$\mathcal{V}^a_o\ ,\ \mathcal{V}^a_1\ ,\ \mathcal{V}^a_2.$$

More explicitely let $[Y] \in \overline{\overline{\mathcal{C}}}$, then the theorem says that $[Y] \in \mathcal{A}$ iff $Y$ is limit of an irreducible, flat family of 1-nodal cubic fourfolds $X$ such that

- $S = X^{\perp} \cap \mathbb{G}$ *is a* 1*-nodal K*3 *surface,*
- *$X$ contains a cone over a rational normal quartic curve,*
- *$S_o$ contains a rational normal quartic,*

## 5.5 Proving A: the rational normal quartic cone

Let $[X] \in \mathcal{A}$ be *general* then Theorem **A** follows if we prove that

$$[X] \in \mathcal{V}^a_0 \cap \mathcal{V}^a_1 \cap \mathcal{V}^a_2.$$

Since $X$ is general in $\mathcal{A}$ we can assume that $X$ is a 1-nodal Pfaffian

$$X = \mathbb{P}^5 \cap \mathbb{D}$$

which is singular at a point $o \in \mathbb{D} - \operatorname{Sing} \mathbb{D}$. Then we have $o = [\mathsf{a}]$, where

$$\mathsf{a} : V \times V \to \mathbb{C}$$

is *an alternating form of rank* 4. Hence Ker $\mathsf{a}$ defines a line in $\mathbb{P}(V)$ that is a point $p = [\mathsf{p}] \in \mathbb{G}$, where $\mathsf{p} = \bigwedge^2 \operatorname{Ker} \mathsf{a}$. We have seen in chapter 5 that the tangent hyperplane to $\mathbb{D}$ at $o$ is the hyperplane orthogonal to $p$, that is

$$\{p\}^{\perp} = \{[\mathsf{b}] \in \mathbb{P}^{14} = \mathbb{P}(\bigwedge^4 V) \mid \mathsf{p} \wedge \mathsf{b} = 0\}.$$

For any $p \in \mathbb{G}$ we know that the hyperplane section $\mathbb{D} \cdot \{p\}^{\perp}$ is the set

$$D_p = \{[\mathsf{b}_1 + \mathsf{b}_2] \in \mathbb{D} \mid [\mathsf{b}_i] \ \in \operatorname{Sing} \mathbb{D},\ \operatorname{Ker} \mathsf{b}_i \cap \operatorname{Ker} \mathsf{a} \neq 0,\ i = 1, 2\}.$$

The equality just says that $D_p$, in the hyperplane $\{p\}^{\perp}$, is the secant variety of the codimension 1 Schubert cycle of $\hat{\mathbb{G}}$, defined by the line $p$. We have already described Sing $D_p$. This is the 5-space in $\mathbb{D}$ generated by the family of points $[\mathsf{b}]$ such that $\mathsf{b}$ has rank 2 and Ker $\mathsf{b}$ contains Ker $\mathsf{a}$:

$$\operatorname{Sing} D_p \ = \langle \{[\mathsf{b}] \in \mathbb{D} \mid \operatorname{Ker} \mathsf{a} \subset \operatorname{Ker} \mathsf{b}\} \rangle.$$

Rephrasing from [Ru], let us recall that the *entry locus.*

**Definition 5.5.1.** Let $o$ be any smooth point of $\mathbb{D}$. The entry locus is the intersection of $\hat{\mathbb{G}}$ with the union of the bisecant lines to $\hat{\mathbb{G}}$ passing through $o$.

**Remark 5.5.2.** The entry locus of $o$ is a smooth 4-dimensional quadric parametrizing the 3-spaces of $\mathbb{P}(V)$ containing the line $\mathbb{P}(\ker \mathsf{a})$.

Let $D$ be a hyperplane section of $\mathbb{D}$ then the following property is true.

**Proposition 5.5.3.** *$D$ is singular at $o$ iff it contains the entry locus of $o$.*

Moreover we know from chapter 5 that $D_p$ contains a rational normal quartic cone, uniquely defining $\{p\}^{\perp}$ as its linear span $\langle D_p \rangle$, namely

$$\mathsf{Q}_p = \{[\mathsf{b}] \in \mathbb{P}^{14} \mid \operatorname{Ker} \mathsf{b} \cap \operatorname{Ker} \mathsf{a} \neq 0\}.$$

This is a cone of vertex Sing $D_p$ over the Segre product $\mathbb{P}^1 \times \mathbb{P}^3$. Finally let $\mathbb{P}^8 = X^{\perp}$ be the orthogonal space of $X$ in $\mathbb{P}(\bigwedge^2 V)$ and $S = \mathbb{P}^8 \cdot \mathbb{G}$. From the description in chapter 5 and Beauville-Donagi paper [BD], we have

$$\mathbb{P}^5 = \bigcap_{q \in \mathbb{P}^8} \{q\}^{\perp}$$

and hence

$$X = \bigcap_{q \in \mathbb{P}^8} D_q.$$

In $\mathbb{P}^8 = X^{\perp}$ let us fix 9 linearly independent points including $p$, say

$$p, q_1 \dots q_8.$$

Then their corresponding orthogonal hyperplane sections

$$D_p,\ D_{q_1} \dots D_{q_8}$$

are defined by linearly independent vectors of $\mathrm{H}^0(\mathcal{O}_{\mathbb{D}}(1))$ because the duality pairing $w : \bigwedge^2 V \times \bigwedge^4 V \to \mathbb{C}$ is non degenerate. Of course $X$ is complete intersection of these and the next theorem follows.

**Proposition 5.5.4.** *$X$ contains a cone of vertex $o$ over a rational normal quartic curve. Hence $S_o$ contains a rational normal quartic curve.*

*Proof.* It follows from the previous discussion that the mentioned quartic cone $\mathrm{Q}_p$ is contained in $D_p$ and that its vertex contains $o$. We have

$$V_o := \mathrm{Q}_p \cdot (D_{q_1} \cdots D_{q_8}) = \mathrm{Q}_p \cdot \mathbb{P}^5 \subset X \subset \mathbb{P}^5 \subset \{p\}^\perp.$$

Since $X$ is general we are allowed to replace it moving $\mathbb{P}^5$ in $\{p\}^\perp$ in an irreducible family $\{\mathbb{P}^5_t, t \in T\}$ so that: $o$ is fixed, the intersection $X_t = \mathbb{P}^5_t \cdot \mathrm{Q}_p$ stays 1-nodal and $\mathbb{P}^5_t$ becomes transversal to $\mathrm{Q}_p$. Hence $V_o$ is a cone over a rational normal quartic curve and the statement follows. □

**Corollary 5.5.5.** $[X]$ *belongs to* $\mathcal{V}^a_1 \cap \mathcal{V}^a_2$.

The corollary implies two statements of theorem A, the next property completes its proof. Starting from $o$ let us fix six linearly independent points $o, o_1 \dots o_5$ in $\mathbb{P}^5$. For their dual hyperplanes we then have

$$\{o\}^\perp \cap \{o_1\}^\perp \cap \cdots \cap \{o_5\}^\perp = \mathbb{P}^8.$$

Let us consider the hyperplane sections $G_o = \mathbb{G} \cap \{o\}^\perp$ and

$$G_i = \mathbb{G} \cap \{o_i\}^\perp,\ i = 1 \dots 5.$$

We must keep $o$, hence $p$ and $G_o$, fixed. However we can move the sections $G_1 \dots G_5$ so that $X = S^\perp \cap \mathbb{D}$ stays 1-nodal at $o$ and the intersection

$$S = G_o \cdots G_5.$$

is transversal at each point different from $o$. We can also move $G_1 \dots G_5$ so that their intersection is a smooth threefold at $p$.

**Proposition 5.5.6.** $\{o\}^\perp$ *contains the projective tangent space to* $\mathbb{G}$ *at* $o$.

*Proof.* Keeping our notation we have $o = [\mathrm{a}]$. Fixing a basis $e_1 \dots e_6$ of $V$, we can assume that the parameter point $p \in \mathbb{G}$ of the line $\mathbb{P}(\ker \mathrm{a})$ is $e_1 \wedge e_2$. Then the projective tangent space to $\mathbb{G}$ at $p$ is $\mathbb{P}(e_1 \wedge V + e_2 \wedge V)$. Hence $\{o\}^\perp$ contains it, since $\{o\}^\perp$ is the projectivized Kernel of the linear form $w(e_1 \wedge e_2, \cdot) : \bigwedge^4 V \to \bigwedge^6 V$, defined by the wedge product with $e_1 \wedge e_2$. □

The proposition implies $\operatorname{Sing} S = \{p\}$. Then the assignment of $X$ to $S$ establishes a birational map between the family of 1-nodal 4-dimensional linear sections $X$ of $\mathbb{D}$ and the irreducible family of singular $K3$ sections of $\mathbb{G}$. It follows that, for a general $X$, $S$ is a general 1-nodal $K3$ surface of genus 8 that is $[S] \in \mathcal{V}^a_o$. This completes the proof of theorem **A**.

## 5.6 The geometric side of A: rational quartic scrolls

An attractive geometric counterpart of theorem A arises, considering the family of rational normal quartic scrolls contained in $X$ when $[X] \in \mathcal{A}$. In what follows we assume $[X]$ general in $\mathcal{A}$, not entering the many interesting specializations. Even so we will see directly several geometric events determined by the family of rational normal quartic scrolls of $X$.

Under our assumption and notation $X$ is 1-nodal and $S = X^{\perp} \cdot \mathbb{G}$ is a general 1-nodal $K3$ section. We can also assume that $\operatorname{Sing} S = \{p\}$, with $p = [e_1 \wedge e_2]$. Then we have $\operatorname{Sing} X = \{o\}$ where $o = [\mathsf{a}]$ and $\mathsf{a}$ is an alternating form with Kernel generated by $e_1, e_2$. Let $p' \in S$ then $p'$ defines

$$\mathsf{Q}_{p'} \subset \mathbb{D} \cap \{p'\}^{\perp},$$

the rational normal quartic cone we have already considered, and hence

$$V_{p'} := \mathsf{Q}_{p'} \cdot \mathbb{P}^5 \subset \mathbb{P}^{12} := \{p, p'\}^{\perp},$$

with $\mathbb{P}^5 := \langle X \rangle$. On the other the line $\overline{pp'}$ is contained in $\mathbb{P}^8 := \langle S \rangle = X^{\perp}$.

**Lemma 5.6.1.** *$\overline{pp'}$ is not in $\mathbb{G}$.*

*Proof.* Since $S = \langle S \rangle \cdot \mathbb{G}$, it suffices to show that $\overline{pp'}$ is not in $S$. Assume $\overline{pp'}$ is in $S$ and consider its strict transform $L' \subset S'$ by $\sigma : S' \to S$. Let $R' = \sigma^{-1}(o)$, since $p \in \sigma(L')$ then $L'R' = 1$. Since $L' \sim mH' + nR'$, for some $m, n \in \mathbb{Z}$, it follows $L'R' = -2n$: a contradiction. □

Let $p' = [a \wedge b]$, by the lemma $\overline{pp'}$ is not parametrizing a pencil of lines of $\mathbb{P}(V)$, that is $e_1 \wedge e_2 \wedge a \wedge b \neq 0$. Hence we can assume $a \wedge b = e_3 \wedge e_4$.

**Lemma 5.6.2.** *Let $H \in |\mathcal{O}_{\mathbb{G}}(1)|$ be the Schubert variety defined by $e_1 \wedge e_2 \wedge e_3 \wedge e_4$, then $S$ is not contained in $H$.*

*Proof.* Assume $S \subset H$ then a pencil $|E|$ of quintic elliptic curves exists in $S'$. More precisely the natural ruling of divisors of the Schubert cycle $H$ pullsback by $\sigma : S' \to S$ to a pencil $|E|$ such that $\deg E = 5$ and $E^2 = 0$. We only mention this fact, to be reconsidered later. As in the previous proof, a simple check on $\operatorname{Pic} S'$ excludes the existence of $E$: a contradiction. □

Now we explicitly describe the cone $\mathsf{Q}_p$ in its hyperplane $\{p\}^{\perp}$. The latter is the projectivization in $\mathbb{P}(\bigwedge^4 V)$ of the vector space having basis:

1. $e_1 \wedge e_2 \wedge e_i \wedge e_j$,

2. $e_m \wedge e_i \wedge e_j \wedge e_k$ ,

for $m = 1, 2$ and $3 \leq i < j < k \leq 6$ or $3 \leq i < j \leq 6$. Notice that

$$\{p\}^{\perp} = \mathbb{P}(W_1 \oplus W_2),$$

$W_1$ being generated by the 6 vectors in (1) and $W_2$ by the 8 vectors in (2). Let $E \subset V$ be the space generated by $\{e_3 \dots e_6\}$, then $\mathbb{P}(W_1)$ is a 5-space isomorphic to $\mathbb{P}(\bigwedge^2 E)$ via the map $(e_1 \wedge e_2 \wedge e_i \wedge e_j) \to (e_i \wedge e_j)$. We already know that $\widehat{\mathbb{G}} \cdot \mathbb{P}(W_1)$ is the Grassmannian of lines of $\mathbb{P}(E)$ and that

$$\mathbb{P}(W_1) = \operatorname{Sing} \mathrm{Q}_p.$$

Hence the vertex $o$ of the cone $V_p = \mathbb{P}^5 \cdot \mathrm{Q}_p$ is a point of $\mathbb{P}(W_1) \cap \{p'\}^{\perp}$, which we always assume not in $\widehat{\mathbb{G}}$. It is useful to recall from chapter 5 that $\{o\}^{\perp}$ stays unchanged when $o$ moves in $\mathbb{P}(W_1)$. Passing to the parametric equations of the cone $\mathrm{Q}_p$ in $\mathbb{P}^{14}$, and in the hyperplane $\{p\}^{\perp}$ spanned by it, it follows that these are provided by the following family of vectors:

$$\lambda e_1 \wedge e_2 \wedge f + \mu(a_1 e_1 + a_2 e_2) \wedge s,$$

with $f \in \bigwedge^2 E$ and $s \in \bigwedge^3 E$. It is useful to describe this family explicitly:

$$\lambda \mathrm{e}_{12} \wedge (f_{34}\mathrm{e}_{34} + f_{56}\mathrm{e}_{56} + f_{35}\mathrm{e}_{35} + f_{36}\mathrm{e}_{36} + f_{45}\mathrm{e}_{45} + f_{46}\mathrm{e}_{46}) \; + \\ + \; \mu(a_1\mathrm{e}_1 + a_2\mathrm{e}_2) \wedge \sum_{(l<m<n)\in(3<4<5<6)} s_{lmn}\mathrm{e}_{lmn},$$

where, for our future convenience, we fix since now the notation

$$e_{i_1} \wedge \dots \wedge e_{i_k} := \mathrm{e}_{i_1 \dots i_k}.$$

The coefficients actually define a dominant rational map

$$\phi : \mathbb{P}^1 \times \mathbb{P}^5 \times \mathbb{P}^1 \times \mathbb{P}^3 \to \mathrm{Q}_p \subset \{p\}^{\perp} = \mathbb{P}^{13},$$

sending $(\lambda : \mu) \times (f_{ij}) \times (a_1 : a_2) \times (s_{lmn})$ to the point of the hyperplane $\{p\}^{\perp}$ with assigned coordinates $(x_{ij} : y_{lmn} : z_{lmn})$ such that

$$x_{ij} = \lambda f_{ij} \; , \; y_{lmn} = \mu a_1 s_{lmn} \; , \; z_{lmn} = \mu a_2 s_{lmn}.$$

Notice that the parametric equations of $\mathrm{Q}_{p'}$ are provided in the same way exchanging the suffixes 1 and 2 with 3 and 4. Now let us move to a further element which stays unchanged when $\mathbb{P}^5$ moves in $\{p, p'\}^{\perp}$, namely

$$p' \in \mathbb{P}^5 \cdot \mathbb{G}.$$

In the variety of the 5-spaces in $\{pp'\}^\perp$ which intersect $\mathbb{P}(W_1) \cap \{p'\}^\perp$ at some point, we consider a suitable open neighborhood $U$ of the parameter point of $\mathbb{P}^5 \in \{pp'\}^\perp$. Suitable means that, for a general $u \in U$, the corresponding 5-space $\mathbb{P}^5_u$ satisfies the same open properties and more of the previous $\mathbb{P}^5$. The list follows after some preparation. Let $\mathbb{P}^8_u = X_u^\perp$, we consider

$$X_u = \mathbb{P}^5_u \cdot \mathbb{D} \text{ and } S_u = \mathbb{P}^8_u \cdot \mathbb{G}.$$

The condition $\mathbb{P}^5_u \subset \{p, p'\}^\perp$ implies that $p = [e_1 \wedge e_2]$ and $p' = [e_3 \wedge e_4]$ stay fixed when $S_u$ moves in $\mathbb{G}$. Consequently $o$ moves in $\mathbb{P}(W_1) \cap \{p'\}^\perp$ and then we can assume that $o$ is a general point (not in $\widehat{\mathbb{G}}$) of the family

$$\{[e_1 \wedge e_2 \wedge f] \ \textit{with } f \in \bigwedge^2 E \textit{ and } f \wedge e_3 \wedge e_4 = 0.\}$$

**(1)** Since we have

$$o \in \mathbb{P}(W_1) \cap \mathbb{P}^5_u \subset \mathbb{P}(W_1) \cup \mathbb{P}^5_u \subset \{p\}^\perp$$

it follows from the previous results that $X_u$ is singular at $o$ and that $S_u$ is singular at $p$. Since $X = X_u$ for some $u$ we can assume, as in the case of $X$ and $S$, that $X_u$ is general 1-nodal at $o$ and that $S_u$ is general 1-nodal at $p$.

**(2)** Let $\sigma_u : S'_u \to S_u$ be the minimal desingularization of $S_u$, we can also assume that $\operatorname{Pic} S_u$ has rank 2 and is constructed as $\operatorname{Pic} S$ from $\sigma : S' \to S$.

**(3)** $V_o = \mathbb{P}^5 \cdot \mathrm{Q}_p$ is a cone over a rational normal quartic curve. Since $\mathrm{Q}_p$ is a cone of vertex $\mathbb{P}(W_1)$ over the Segre product $\mathbb{P}^1 \times \mathbb{P}^3$, it follows that $\mathbb{P}^5$ is a general 5-space of $\{pp'\}^\perp$ which is transversal to $\mathrm{Q}_p$ and such that $\{o\} = \mathbb{P}(W_1) \cap \mathrm{Q}_p$. Then the same property occurs for a general $\mathbb{P}^5_u$.

**(4)** Let us continue with our point $o = [e_1 \wedge e_2 \wedge f]$ such that $f \in \bigwedge^2 E$ and

$$f \wedge e_3 \wedge e_4 = 0.$$

As remarked the points of the $Q_{p'}$ are obtained, from the the description of the points of $Q_p$, just exchanging the suffixes 1 and 2 with 3 and 4. This implies that any point of the cone $Q_{p'}$ is represented as follows:

$$\lambda e_3 \wedge e_4 \wedge g + \mu(a_3 e_3 + a_4 e_4) \wedge t,$$

where $g \in \bigwedge^2 G$, $t \in \bigwedge^3 G$ and $G \subset V$ is the space generated by $e_1, e_2, e_5, e_6$.

**Proposition 5.6.3.** *The point $o$ does not belong to $\mathrm{Q}_{p'}$ for a general $u \in U$.*

*Proof.* Since $o$ is in $\mathbb{P}(W_1) = \operatorname{Sing} \mathsf{Q}_p$, the condition to have $o \in \mathsf{Q}_{p'}$ is

$$[\lambda e_3 \wedge e_4 \wedge g + \mu(a_3e_3 + a_4e_4) \wedge t] = [e_1 \wedge e_2 \wedge f].$$

Inspecting the vectors it follows that $t = e_1 \wedge e_2 \wedge (a_5e_5 + a_6e_6)$ and that $g = e_1 \wedge e_2$. Then $o$ is in the following set of points of $\mathbb{P}(W_1) \cap \mathsf{Q}_{p'}$:

$$[e_1 \wedge e_2 \wedge (\lambda(a_3e_3 + a_4e_4) \wedge (a_5e_5 + a_6e_6)) + \mu e_3 \wedge e_4)] \in \mathbb{P}(W_1).$$

This is a quadric cone of vertex $[e_1 \wedge e_2 \wedge e_3 \wedge e_4]$ in the 4-space

$$\{f_{56} = 0\} = \{p'\}^{\perp} \cap \mathbb{P}(W_1)$$

Hence a general $o$ in this 4-space does not belong to $\mathsf{Q}_{p'}$. □

**(5)** The previous proof helps to easily describe the intersection

$$\mathsf{Q}_{p'} \cdot \{p, p'\}^{\perp}.$$

Since $\mathsf{Q}_{p'} \subset \{p'\}^{\perp}$, we have only to determine $\mathsf{Q}_{p'} \cap \{p\}^{\perp}$. As observed the parametric equations of $\mathsf{Q}_{p'}$ are obtained from those of $\mathsf{Q}_p$ exchanging $1, 2$ and $3, 4$. Hence $\mathsf{Q}_{p'}$ is the projectivizated set of vectors

$$\begin{aligned}[][\lambda \mathsf{e}_{34} \wedge (g_{12}\mathsf{e}_{12} + g_{56}\mathsf{e}_{56} + g_{15}\mathsf{e}_{15} + g_{16}\mathsf{e}_{16} + g_{25}\mathsf{e}_{25} + g_{26}\mathsf{e}_{26}) \ + \ \mu(a_3\mathsf{e}_3 + a_4\mathsf{e}_4) \\ \wedge \sum_{(l<m<n)\in(1<2<5<6)} t_{lmn}\mathsf{e}_{lmn}] \ = \ [\lambda e_3 \wedge e_4 \wedge g + \mu(a_3e_3 + a_4e_4) \wedge t].\end{aligned}$$

For $\mu = 0$ we obtain the 5-space $\operatorname{Sing} \mathsf{Q}_{p'} = \mathbb{P}(\bigwedge^2 G)$. For $\lambda = 0$ we obtain the product $\mathbb{P}^1 \times \mathbb{P}^3$ with coordinates $(a_3 : a_4) \times (t_{124} : t_{125} : t_{145} : t_{245})$. Now, working exactly as above we obtain, after the exchange of suffixes, that

$$\operatorname{Sing} \mathsf{Q}_{p'} \cap \{p\}^{\perp} = \{g_{56} = \mu = 0\}$$

in other words we obtain the set of points

$$\{[\mathsf{e}_{34} \wedge (g_{12}\mathsf{e}_{12} + g_{15}\mathsf{e}_{15} + g_{16}\mathsf{e}_{16} + g_{25}\mathsf{e}_{25} + g_{26}\mathsf{e}_{26})]\}.$$

Then the next property easily follows, we omit further details.

**Proposition 5.6.4.** *$\mathsf{Q}_{p'} \cdot \{pp'\}^{\perp}$ is a cone over $\mathbb{P}^1 \times \mathbb{P}^3$ whose vertex is the linear space* $\operatorname{Sing} \mathsf{Q}_{p'} \cap \{p\}^{\perp}$.

**Definition 5.6.5.** *For the above section of* $\mathrm{Q}_{p'}$ *we define*

$$\mathrm{Q}_{p'} \cdot \{pp'\}^{\perp} := \mathrm{Q}^{p}_{p'}.$$

The definition works for any pair $(p, p') \in \mathbb{G} \times \mathbb{G}$ such that $p \neq p'$. Now we start to profit of the previous *effective description* to deduce some useful properties of the family $\{X_u,\ u \in U\}$. At first we point out that $\mathrm{Q}^p_{p'}$ is a quartic cone in $\mathbb{P}^{12} = \{pp'\}^{\perp}$ over a smooth variety with singular locus the 4-space $\operatorname{Sing} \mathrm{Q}_{p'} \cap \{p\}^{\perp}$. By the previous results we can assume that $o$ is not in $\mathrm{Q}_{p'}$ and keep it fixed moving $\mathbb{P}^5_u$ in $\mathbb{P}^{12} = \{pp'\}^{\perp}$. $\mathbb{P}^5_u$ is indeed a general 5-space through $o$. By dimension count it is disjoint from $\operatorname{Sing} \mathrm{Q}^p_{p'}$ and it is transversal to $\mathrm{Q}^p_{p'}$ by Bertini theorem. This implies the next property.

**Proposition 5.6.6.** $Q^p_{p'} \cdot \mathbb{P}^5_u$ *is a smooth rational quartic scroll for a general* $u$*, which is not passing through* $o$*.*

**(6)** Finally we go back to the family of quartic scrolls $\{V_{p'}\ p' \in S\}$ we have already considered. Since we move $\mathbb{P}^5$ in a family $\{\mathbb{P}^5_u, u \in U\}$, we will also consider the induced families of scrolls

$$\{V_{p'}(u), p' \in S_u\},$$

parametrized by the family of surfaces $S_u$, which are 1-nodal K3 sections of $\mathbb{G}$ at $o = o(u) \in \operatorname{Sing} \mathrm{Q}_p$, for a general $u \in U$. It is the moment to summarize some well known properties of any smooth quartic scroll

$$V \subset \mathbb{P}^5.$$

**Proposition 5.6.7.** $V$ *is projectively normal and its ideal is generated by quadrics.*

$V$ is a very well known and interesting surface, specially from the point of view of the variety of its bisecant lines. Indeed $V$ is one of the two possible cases of smooth OADP surfaces, see [CR]. This means that, through a general point $x \in \mathbb{P}^5$, a unique bisecant line to $V$ is passing.

**Definition 5.6.8.** *Let* $x \in \mathbb{P}^5$*, we say that* $x$ *is in general position with* $V$ *if a unique bisecant line to* $V$ *is passing through* $x$*.*

In the opposite case it is well known that then $x$ is in a plane containing a conic of $V$, cfr. [CR]. Let $L$ be a bisecant line to $V$, not contained in $V$. Notice also that, since $V$ is generated by quadrics, the intersection scheme

$$\zeta = L \cdot V$$

has length 2. We are in the position to claim that:

*Every bisecant line to $V$, not contained in $V$, is exactly bisecant to $V$.*

**Lemma 5.6.9.** *$o$ is in general position with respect to $V_{p'}(u)$ for a general $u \in U$.*

*Proof.* We know from the study of $\operatorname{Pic} S_o$ that no line is in $S_o$. Assume $o$ is not in general position with respect to $V_{p'}$. Then $o$ is in a plane $\Pi$ containing a conic $C$ of $V_p$. Since $o$ is not in $V_{p'}$, $\Pi$ contains two conics with no common components: $C$ and a conic singular at $o$. Hence $\Pi$ is in the cone $F(X)_o = \{F_2 = F_3 = 0\}$ and its projection $\pi_o(\Pi)$ is a line in $S_o$: a contradiction. This shows the statement for $X$. Since the condition of being in general position is open, the same follows for $X_u$, $u$ general in $U$. □

**(7)** We add to our usual notation the following: if $o$ is in general position with $V_{p'}$ then $\ell_{p'}$ is exactly the unique exactly bisecant line to $V_{p'}$ and

$$\zeta_{p'} = \ell_{p'} \cdot V_{p'}.$$

Then $\zeta_{p'}$ is a scheme of length two, consisting either of two distinct points or of one subscheme of multiplicity two in $\ell_{p'}$. We fix since now the projection

$$\pi_o : \mathbb{P}^5 \to \mathbb{P}^4$$

and denote as usual the equation of $X$ as $t_6 F_2 + F_3$ so that

$$F(X)_o = \{F_2 = F_3 = 0\}$$

is the cone over the K3 surface $S_o = \pi_o(F(X)_o)$. We fix the notation

$$Q_2 = \pi_o(\{F_2 = 0\}) \subset \mathbb{P}^4 \,,\; Q_3 = \pi_o(\{F_3 = 0\}) \subset \mathbb{P}^4$$

so that $S_o = Q_2 \cap Q_3$. The next lemma follows from Bèzout theorem.

**Lemma 5.6.10.** *$\ell_p$ is contained in the cubic fourfold $X$.*

Let us also define the following curve for a general $p' \in S$:

**Definition 5.6.11.** *The hyperelliptic curve associated to $p'$ is*

$$\hat{B}_{p'} := \{F_2 = 0\} \cdot V_{p'}.$$

$\hat{B}_{p'}$ is a curve: $F_2$ cannot vanish on $V_{p'}$. Otherwise $\pi_o(V_{p'})$ would be a component of $S_o$: a contradiction. $\hat{B}_{p'}$ is cut on $V_{p'}$ by the quadratic projective tangent cone of $X$ at $o$: independently from the choice of coordinates.

**Lemma 5.6.12.** *$\hat{B}_{p'}$ is hyperelliptic of arithmetic genus $p_a(\hat{B}_{p'}) = 3$ and degree 8.*

*Proof.* Computing $\deg(\hat{B}_{p'})$ and $p_a(\hat{B}_{p'})$ in $V_{p'}$ is standard. Since $\hat{B}_{p'}$ is a quadratic section, the unique ruling of lines $|F|$ of $V_{p'}$ of $V_{p'}$ defines a line bundle $\mathcal{O}_{\hat{B}_{p'}}(F)$ on $\hat{B}_{p'}$ and a finite map of degree two $f : \hat{B}_{p'} \to \mathbb{P}^1$. □

The projection $\pi_o$ restricts to a morphism

$$\pi_o : \hat{B}_{p'} \to S_o \subset \mathbb{P}^4.$$

This follows because $\hat{B}_{p'}$ is contained in $\{F_2 = 0\}$ by definition and we also have $\hat{B}_{p'} \subset V_{p'} \subset X = \{t_6 F_2 + F_3 = 0\}$. This implies $\pi_{o*}\hat{B}_o \subset Q_2 \cap Q_3 = S_o$.

**Definition 5.6.13.** *The quasi hyperelliptic curve of $p'$ is $B_{p'} = \pi_{o*}\hat{B}_{p'}$.*

We will say that a curve is *quasi hyperelliptic* at a regular point $t$ if a partial normalization at $t$ is hyperelliptic. Now we want to point out that

$$\hat{B}_{p'} \cdot \ell_{p'} = \zeta_{p'},$$

where $\ell_{p'}$ is the unique exactly bisecant line to $V_{p'}$ from $o$. This follows because, as a line of $X$ through $o$, $\ell_{p'}$ is contained in $\{F_2 = 0\}$. Hence $\pi_o|\hat{B}_{p'}$ contracts exactly $\zeta_{p'} \subset \hat{B}_{p'}$. For the rest the map is biregular, since the same is true for $\pi_o|V_{p'}$. This implies the next lemma.

**Lemma 5.6.14.** *$B_{p'}$ is quasi hyperelliptic of arithmetic genus 4 and degree 8.*

Let us now introduce some important properties of the linear system of genus four curves defined by the curve $B_{p'}$ on $S_o$, let us denote it

$$|B|.$$

**Lemma 5.6.15.** $|B| = |2H_o - R_o|$.

*Proof.* Let $\lambda : \mathbb{P}^5 \to \mathbb{P}^3$ be the projection from the bisecant line $\ell_{p'}$ of $V_{p'}$. Since $\ell_{p'} \cdot V_{p'}$ has length 2 then $\lambda(V_{p'})$ is an integral quadric surface. Hence Its inverse image by $\lambda$ is a quadric hypersurface $\hat{Q} \subset \mathbb{P}^5$. $\hat{Q}$ is singular along $\ell_{p'}$ and contains $V_{p'}$. Then it follows that $Q = \pi_o(\hat{Q})$ is a quadric containing $B_{p'}$ and not $S_o$. This implies that $|2H_o - B_{p'}|$ contains an effective curve $R$ such that $H_o R = 4$ and $R^2 = -2$. Since $R \sim xH_o + yR_o$ one computes that then $x = 0, y = 1$ so that $R = R_o$. □

**Lemma 5.6.16.**

*(1) Every element of $|B|$ is an integral curve.*

*(1) $|B|$ is not a hyperelliptic linear system.*

*Proof.*

(1) Assume $B \in |2H_o - R_o|$ is not integral and properly contains an integral $D$, then $D \sim xH_o + yR_o$ so that $0 < H_oD = 6x + 4y < 8$ and $R_oD = 4x - 2y$. This implies $0 < 7x < 4$ so that $x$ is not an integer.

(2) Assume $|B|$ is hyperelliptic. As it is well known then $S_o$ contains a pencil $|E|$ of elliptic curves such that $EB = 2$, [Huy1]. But then $E^2 = 0$ and $\operatorname{Pic} S_o$ contains an isotropic vector, which is readily excluded by computing in $\operatorname{Pic} S_o$.

□

**Theorem 5.6.17.** *For each $t \in S_o$ there exists exactly one $B_t \in |B|$ such that $t$ is an ordinary double point of $B_t$ and $B_t$ is quasi hyperelliptic at $t$.*

*Proof.* Let us consider the point $z = \pi_o(\zeta_{p'})$ and its ideal sheaf $\mathcal{I}_z$ in $S_o$. Since $|B|$ is not hyperelliptic it follows that $|\mathcal{I}_z(B)|$ has a unique and simple base point, which is $z$. Since $\dim |B| = 4$, $|\mathcal{I}_z(B)|$ defines a rational map

$$\phi_z : S_o \to \mathbb{P}^3.$$

Since $z$ is simple and $B^2 = 6$ then $\phi_z$ is birational onto its image and this is an integral quintic surface $S_z = \phi_z(S_o)$. Moreover, a general $B \in |\mathcal{I}_z(B)|$ is smooth of genus 4 and $\phi_z(B)$ is a general plane section of $S_z$. Then this curve, as an integral plane quintic of geometric genus 4, has no singular point of multiplicity $\geq 3$ and only double points: two distinct ordinary nodes or cusps or a tacnode counting for two. This implies that $\operatorname{Sing} S_z$ is a curve of degree 2. Assume $\operatorname{Sing} S_z$ is a conic and let $\Pi \subset \mathbb{P}^3$ be its plane, then $\Pi \cdot S_z = 2 \operatorname{Sing} S_z + E$, where $E$ is a line. Precisely $\phi_z$ factors as $\phi_z = \phi' \circ \sigma^{-1}$, where $\sigma : S'_o \to S_o$ is the blowing up of $S_o$ at $z$ and $\phi'$ is a morphism. Then $E$ is the image by $\phi'$ of the exceptional line $E_z := \sigma^{-1}(z)$. Now let us consider the pull-back of $\Pi$ by $\phi_z$. This is a curve $B_z \in |B|$. Since $B_z$ is integral then $\operatorname{Sing} S_z$ is a smooth conic. Also, $B_z$ is hyperelliptic because $\phi_z : B_z \to \operatorname{Sing} S_z$ has degree two. It also follows that the multiplicity of $B_z$ at $z$ is two. This implies the existence and uniqueness of $B_z$, when $\operatorname{Sing} S_z$ is a conic. Otherwise $\operatorname{Sing} S_z$ is union of two disjoint double lines of $S_z$. But then the pencil of planes through any of these lines cuts on $S_z$ a pencil of plane cubics. This lifts by $\phi_z$ to an elliptic pencil on $S_o$: a contradiction. □

Using the latter theorem and the preceeding results we finally define our favourite rational map.

**Definition 5.6.18.** *Let $p' \in S$ then $f : S \dashrightarrow S_o$ is the rational map such that*

$$f(p') = z := \pi_o(\zeta_{p'}).$$

For the moment $f$ is just some rational map, the next theorem makes clear what happens and completes the picture of the geometric side of **A**.

**Theorem 5.6.19.** *$f$ lifts to a biregular map $f' : S' \to S_o$ moreover one has*

$$\sigma^* \mathcal{O}_S(1) \cong \mathcal{O}_{S_o}(H_o + 2R_o).$$

*Proof.* Let us consider as usual a general $p' \in S$ and $z = f(p')$. To show that $f$ is birational it suffices to reconstruct the quartic scroll $V_{p'}$, and hence $p'$, from $z$. Recall that $z = \pi_o(\zeta_{p'})$ and that $B_{p'} = \pi_{o*}\hat{B}_{p'}$ passes through $z$, where $\hat{B}_{p'} = V_{p'} \cdot \{F_2 = 0\}$. We know that $B_{p'} \in |B|$ is quasi hyperelliptic at its singular point $z$. By the previous theorem these properties characterize a unique curve $B_z \in |\mathcal{I}_z(B)|$, described in its proof. Hence we have $B_{p'} = B_z$ and $B_{p'}$ is uniquely defined just starting from the point $z$. Then the reconstruction of $V_{p'}$ from such a curve is not difficult. At first we reconstruct $\pi_o(V_{p'})$: let $\iota : B_z \dashrightarrow B_z$ be the birational hyperelliptic involution of $B_z$. In fact $\pi_o(V_{p'})$ is equal to the union of lines defined as

$$V_z := \bigcup_{y \in B_z} \overline{y\iota(y)}.$$

Then $V_{p'}$ is reconstructed as the image of $V_z$ via $\pi_o^{-1} : \mathbb{P}^4 \dashrightarrow X$. Hence the construction defines a rational section $g : S_o \dashrightarrow S$ of $f$, which is sending a general $z \in S_o$ to $p'$. Since $S$ and $S_o$ are irreducible of the same dimension then $g$ and $f$ are inverse one to the other, hence $f, g$ are birational. Passing to the minimal desingularization $S'$ of $S$ the birational map $f$ lifts to a biregular map $f' : S' \to S_o$ by Castelnuovo theorem. To complete the proof note that $\operatorname{Pic} S_i$ has a unique effective class of degree 14. Since $H_o + 2R_o$ defines an effective class of degree 14, then $\sigma^* \mathcal{O}_S(1) \cong \mathcal{O}_{S_o}(H_o + 2R_o)$. $\square$

## 5.7 Rationality of $\mathcal{A}$

As we saw, $\mathcal{A}$ is birational to the moduli space of K3 of degree 6 containing a rational normal quartic curve. Since a general such K3 contains only one rational normal quartic, then a birational model of $\mathcal{A}$ is, for a fixed rational normal quartic $C$, is

$$\mathcal{F}_C = \{[S] \in \mathcal{F}_4 | S \supset C\}.$$

Let $\mathsf{Q}_C$ be the Hilbert scheme of quadric threefolds containg $C$ and $\mathsf{S}_C$ be the Hilbert scheme of K3 surfaces of degree 6 containing $C$. Then there is a natural map

$$\pi : \mathsf{S}_C \to \mathsf{Q}_C,$$

which associates a K3 surface $S$ to the unique quadric hypersurface containing it. $\pi^{-1}(Q)$ is naturally identified with $\mathbb{P}(\mathrm{H}^0(\mathcal{I}_{C/Q}(3))) \cong \mathbb{P}^{16}$, so $\mathsf{S}_C$ is indeed a sub-bundle of the projectivized anti-canonical bundle over $\mathsf{Q}_C$. The map $\pi$ is compatible with the action of $\mathrm{Aut}(C)$ and the stabilizer in $\mathrm{Aut}(C)$ of the general $Q \in \mathsf{Q}_C$ is trivial. It follows that $\pi$ descends to a $\mathbb{P}^{16}$-bundle

$$\bar{\pi} : \mathcal{S}_C \to \mathcal{Q}_C.$$

$\mathcal{Q}_C$ is unirational and 2-dimensional so, by Castelnuovo theorem, is rational. The rationality of $\mathcal{S}_C$ follows immediately.

## 5.8 Stating theorem B

In this case the key words for defining $\mathcal{V}$ are:

*elliptic K3 surfaces of genus 4 and 8 , cone in $\mathbb{P}^5$ over a quintic elliptic curve.*

Let us fix a point $\mathsf{o} \in \widehat{\mathbb{G}} \subset \mathbb{D}$, $L \subset \mathbb{P}^{14}$ a 5-dimensional projective subspace containing $\mathsf{o}$, $X = \mathbb{D} \cap L$. We can finally state Theorem B:

**Theorem 5.8.1** (Theorem B). *A general $X \in \mathcal{B}$ has the following properties:*

1. *$S = X^{\perp} \cap \mathbb{G}$ contains a 1-dimensional family of quintic elliptic curves;*
2. *$S_o$ contains a pencil of quintic elliptic curves;*
3. *$X$ contains a pencil family of cones over a quintic elliptic curve.*

## 5.9 Proving theorem B

*Proof of* B.1.

Write $\mathsf{o} := [v_1 \wedge v_2 \wedge v_3 \wedge v_4]$ and $V_{\mathsf{o}} := \mathrm{Span}(\{v_1, \dots, v_4\})$.

Let $H \subset V$ be a hyperplane containing $V_{\mathrm{o}}$. Then $G(2,H) \cap S = G(2,H) \cap L^{\perp}$ is a divisor of $S$: in fact both $L^{\perp}$ and $G(2,H)$ are contained in the hyperplane $\mathrm{o}^{\perp}$, so $\dim(G(2,H) \cap L^{\perp}) = \dim(G(2,H)) - \mathrm{codim}_{\mathrm{o}^{\perp}}(L^{\perp}) = 6-5 = 1$ for a general choice of $L$.

$D_H := G(2,H) \cap L^{\perp}$ is in a general smooth and irreducibe divisor. Note that two general distinct $D_{H_1}, D_{H_2}$ are disjoint, since $G(2,H_1) \cap G(2,H_2) = G(2,V_{\mathrm{o}})$, which has dimension 4. They are cohomologically equivalent (they correspond to the same Schubert cycle), so $D_H^2 = 0$ in $\mathrm{Pic}(S)$. This implies that $D_H$ is smooth of genus 1. Since $\deg(G(2,5)) = 5$, it follows that $\deg(D_H) = 5$. □

Before proving B.2 let us introduce some preliminaries results.

**Lemma 5.9.1.**

*(1) Let $L \subset \widehat{\mathbb{G}}$ be a 4-dimensional linear subspace of $\mathbb{P}(\bigwedge^2 V)$. Denote by $\pi_L$ the projection from the subspace K. Then*

$$\pi_L(\widehat{\mathbb{G}}) \cong G(2,5).$$

*(2)*

$$\pi_{T_{\mathrm{o}}\mathbb{G}}(\mathbb{G}) \cong G(2,4).$$

*Proof.*

(1) The 4-dimensional projective subspaces contained in $\mathbb{G}$ are all of the form
$$L_{\mathrm{p}} = \{\ell \in \mathbb{G} : \mathrm{p} \in \ell\},$$
where $\mathrm{p} \in \mathbb{P}(V)$.
We can assume, without loss of generality, that $V = \mathbb{C}^6, \mathrm{p} = [\mathrm{e}_1]$. In this case
$$L_{\mathrm{p}} = \mathbb{P}(\mathrm{Span}(\mathrm{e}_1 \wedge \mathrm{e}_2, ..., \mathrm{e}_1 \wedge \mathrm{e}_6)).$$
$\pi_{L_{\mathrm{p}}}$ is identified with the quotient map
$$\mathbb{P}(\bigwedge^2 V) \to \mathbb{P}(\bigwedge^2 (V/\langle \mathrm{e}_1 \rangle)),$$
which maps $\mathbb{G} = G(2,V)$ to $G(2, V/\langle \mathrm{e}_1 \rangle)$.

(2) Assuming that $V = \mathbb{C}^6$ and $\mathrm{o} = \mathrm{e}_1 \wedge \mathrm{e}_2$, we have that
$$T_{\mathrm{o}}\mathbb{G} = \mathbb{P}(\mathrm{Span}(\{\mathrm{e}_1 \wedge \mathrm{e}_i, \mathrm{e}_2 \wedge \mathrm{e}_j\}_{\substack{2\le i\le 6\\ 3\le j\le 6}})).$$

So $\pi_{T_{\mathrm{o}}\mathrm{G}}$ is actually the natural projection map

$$\mathbb{P}(\bigwedge^2 V) \to \mathbb{P}(\bigwedge^2 (V/\langle \mathsf{e}_1, \mathsf{e}_2 \rangle)),$$

which maps $\mathbb{G} = G(2,V)$ to $G(2, V/\langle \mathsf{e}_1, \mathsf{e}_2 \rangle)$.

$\square$

Before stating the next result, we recall the notion of join.

**Definition 5.9.2.** Let $Y, Z \subset \mathbb{P}^n$ two embedded projective varieties. The join of $Y$ and $Z$, denoted $\mathrm{J}(Y,Z)$, is the variety

$$\overline{\bigcup_{y \in Y, z \in Z} \ell_{y,z}}.$$

**Corollary 5.9.3.**

*(1) Let $L \subset \mathbb{P}(\bigwedge^2 V)$ be a 4-dimensional linear subspace contained in $\mathbb{G}$. Then $\mathrm{J}(L,\mathbb{G})$ is a cone of vertex $L$ over $G(2,5)$. In particular $\mathrm{J}(L,\mathbb{G})$ is 11-dimensional and of degree 5.*

*(2) $\mathrm{C}_{\mathrm{o}}\widehat{\mathbb{D}} = \mathrm{J}(\mathrm{T}_{\mathrm{o}}\mathbb{G}, \mathbb{G})$. Then it is a cone of vertex $T_{\mathrm{o}}\mathbb{G}$ over $G(2,4)$.*

*Proof.*

(1) In fact $\mathrm{J}(L,\mathbb{G}) = \pi_L^{-1}(\pi_L(\mathbb{G})) = \pi_L^{-1}(G(2,5))$.

(2) It is sufficient to prove the inclusion $\supseteq$, since both of them are quadric hypersurfaces. Note that $T_{\mathrm{o}}\mathbb{G} = \mathrm{Sing}(C_{\mathrm{o}}\widehat{\mathbb{D}})$ and $\mathbb{G} \subset C_{\mathrm{o}}\widehat{\mathbb{D}}$. In particular the line joining a point of $T_{\mathrm{o}}\mathbb{G}$ and a point of $\widehat{\mathbb{D}}$ is contained in $C_{\mathrm{o}}\widehat{\mathbb{D}}$. This proves the equality. The other statement follows from the chain of equalities:

$$\mathrm{J}(T_{\mathrm{o}}\mathbb{G}, \mathbb{G}) = \pi_{T_{\mathrm{o}}\mathrm{G}}^{-1}(\pi_{T_{\mathrm{o}}\mathrm{G}}(\mathbb{G})) = \pi_{T_{\mathrm{o}}\mathrm{G}}^{-1}(G(2,4)).$$

$\square$

**Corollary 5.9.4.** *Denote by $\ell_{\mathsf{o}} \subset \mathbb{P}(V)$ the straight line representing* $\mathsf{o}$. *Then*

$$\mathrm{C}_{\mathsf{o}}\widehat{\mathbb{D}} \cap \widehat{\mathbb{D}} = \bigcup_{\mathsf{p} \in \ell_{\mathsf{o}}} \mathrm{J}(\mathrm{L}_{\mathsf{p}}, \mathbb{G}) = \mathrm{J}(\mathrm{T}_{\mathrm{o}}\mathbb{G} \cap \mathbb{G}, \mathbb{G}).$$

*Proof.* The second equality is obvious from the definition. Since $\widehat{\mathbb{D}} = \mathrm{J}(\mathbb{G}, \mathbb{G})$ by definition and $C_{\mathrm{o}}\widehat{\mathbb{D}} = J(T_o\mathbb{G}, \mathbb{G})$ by the previous result, the inclusion $C_{\mathrm{o}}\widehat{\mathbb{D}} \cap \widehat{\mathbb{D}} \supseteq J(T_o\mathbb{G} \cap \mathbb{G})$ is clear. Since the left side is 12-dimensional and irreducible and the right side is 12-dimensional (being the union of a pencil of 11-dimensional cones), the equality follows. $\square$

Recall that the K3 surface (of degree 6) associated to a cubic fourfold $X$ singular in $\mathsf{x} \in X$ is $\pi_{\mathsf{x}}(X \cap C_{\mathsf{x}}X)$, where $\pi_{\mathsf{x}}$ denotes the projection from $\mathsf{x}$.

In our case, if $L$ is a general 5-dimensional projective space passing through $\mathsf{o}$ and $X = \widehat{\mathbb{D}} \cap L$, the associated K3 surface is $\pi_{\mathsf{o}}(\widehat{\mathbb{D}} \cap C_{\mathsf{o}}\widehat{\mathbb{D}} \cap L)$.

*Proof of* B.2 *and* B.3. Let $L$ be a general 5-dimensional projective space passing through $\mathsf{o}$. If $L$ is general, then $L \cap T_{\mathsf{o}}\mathbb{G} = \{\mathsf{o}\}$. It follows that $L \cap C_{\mathsf{o}}\widehat{\mathbb{D}}$ is a cone of vertex $\mathsf{o}$ over a quintic curve of $\mathbb{P}^4$, which is smooth for the assumption of generality on $L$. This proves B.3. Recall that $S_{\mathsf{o}} = \pi_{\mathsf{o}}(\widehat{\mathbb{D}} \cap L \cap C_{\mathsf{o}}\widehat{\mathbb{D}})$. In particular $S_{\mathsf{o}}$ contains the pencil of quintic elliptic curves $\{\pi_{\mathsf{o}}(\mathrm{J}(L_{\mathsf{p}}, \mathbb{G}) \cap L)\}_{\mathsf{p}} \in \ell_{\mathsf{o}}$. $\square$

The K3 surfaces $S_{\mathsf{o}}$ and $X^{\perp} \cap \mathbb{G}$ are indeed isomorphic, as explained by the next result.

**Proposition 5.9.5.** *Every elliptic K3 surface of degree 6 (14) containing a family of quintic elliptic curves admits a primitive polarization of degree 14 (6). Also this polarization contains a family of quintic elliptic curves.*

*Proof.* Suppose that $\deg(S) = 14$. Let $h$ be the cohomology class of the hyperplane section, $c$ the class of the quintic curve. The following equalities hold:

$$h^2 = 14, h \cdot c = 5, c^2 = 0.$$

Define $l := 3h - 4c$. Then $l^2 = 6$, so $l$ gives a primitive polarization of degree 6. Define $\gamma := 5h - 7c$, then

$$\gamma^2 = 0, \gamma \cdot l = 5.$$

So the curves whose cohomology class is $\gamma$ are elliptic of degree 5 in the polarization induced by $l$. Since the matrix $\begin{pmatrix} 3 & -4 \\ 5 & -7 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$, then also the inverse statement holds. $\square$

# Chapter 6

# The universal K3 of genus 8 is rational

In this chapter we describe the proof of the rationality of $\mathcal{F}_{8,1}$ in [DiT].

## 6.1 $\mathcal{F}_{8,1}$ as $\mathbb{P}^{16}$-bundle over $\mathrm{Pic}_{3,2}$

Denote by

$$\tilde{\mathcal{C}}_{14} := \{(X, R) : X \in |\mathcal{O}_{\mathbb{P}^5}(3)|, R \subset X \text{ is a quartic scroll}\} /\!/ \mathrm{PGL}(6)$$

the incidence variety of quartic scrolls and smooth cubic fourfolds. This variety is birational to $\mathcal{F}_{8,1}$. So proving the rationality of the universal K3 of genus 8 is equivalent to prove the rationality of $\tilde{\mathcal{C}}_{14}$ The main idea of the proof is to show that $\tilde{\mathcal{C}}_{14}$ has a structure of projective bundle over $\mathrm{Pic}_{3,2}$. Here we use the notation $\mathrm{Pic}_{d,g}$ to denote the universal Picard variety of line bundles of degree $d$ over a curve of genus $g$. In what follows this is the coarse moduli space of pairs $(C, \mathcal{L})$ such that $C$ is a smooth integral curve of genus $g$ and $\mathcal{L} \in \mathrm{Pic}_d(C)$, see [HM] for the main general properties and definitions. In our case a birationally equivalent construction of it as a GIT quotient can be provided as follows. Observe that a pair $(C, \mathcal{L})$, defining a general point of $\mathrm{Pic}_{3,2}$, provides an embedding

$$C \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1$$

as a curve of type $(3, 2)$ such that $\mathcal{L} \cong \mathcal{O}_C(0, 1)$ and $\omega_C \cong \mathcal{O}_C(1, 0)$. Then a birationally equivalent model of $\mathrm{Pic}_{3,2}$ is the GIT quotient

$$|\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(3, 2)| /\!/ \mathrm{Aut}(\mathbb{P}^1)^2$$

Let $R \subset \mathbb{P}^5$ be a quartic scroll. The moduli space $\tilde{\mathcal{C}}_{14}$ is also described as a quotient $\mathfrak{C}_R / G_R$ where

$$\mathfrak{C}_R := \{X \in |\mathcal{O}_{\mathbb{P}^5}(3)| \text{ s.t. } X \supset R\} \text{ and } G_R := \{f \in \mathrm{PGL}(5) \text{ s.t. } f(R) = R\}$$

This is clear since every general smooth quartic scroll can be moved to a fixed one by an automorphism of $\mathbb{P}^5$ . The first step is to construct a projective bundle

$$\mathfrak{C}_R \to \left|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(3,2)\right|$$

We recall that a Segre product is the embedding of $\mathbb{P}^a \times \mathbb{P}^b$ defined by the line bundle $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(1,1)$, up to projective automorphisms. The degree of this embedding is $\binom{a+b}{a}$. Clearly a Segre product of degree 3 is $\mathbb{P}^1 \times \mathbb{P}^2$ embedded in $\mathbb{P}^5$. For convenience in the exposition we will say that

**Definition 6.1.1.** A cubic Segre product is an embedding $\mathbb{P}^1 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$ as above.

Note that $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(1,0)|$ is the pencil of respectively disjoint planes

$$\{x\} \times \mathbb{P}^2, x \in \mathbb{P}^1.$$

On the other hand the elements of $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(0,1)|$ are smooth quadric surfaces, namely products $\mathbb{P}^1 \times \mathcal{L}$ with $\mathcal{L} \in \mathcal{O}_{\mathbb{P}^2}(1)$. We are interested in isomorphic copies of smooth quadric surfaces of degree 4 in $\mathbb{P}^1 \times \mathbb{P}^2$. To this purpose let us point out that the surfaces of degree 4 in $\mathbb{P}^1 \times \mathbb{P}^2$ are distributed in two linear systems:

- $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(0,2)|$;
- $\left|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(2,1)\right|$.

The next propositions describe smooth quartic scrolls from these linear systems. We fix the usual notation $\mathbb{F}_n$ for the $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ with minimal section $e$ of self intersection $-n$. We denote the fibre of $\mathbb{F}_n \to \mathbb{P}^1$ by $f$. As is well known $\mathbb{F}_n$ is a Hirzebruch surface. We are interested to $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ and to $\mathbb{F}_2$ which is a rank 3 quadric cone blown up at its vertex.

**Proposition 6.1.2.** *Let $R \in \left|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(0,2)\right|$ be a smooth and irreducible divisor. Then $R \cong \mathbb{F}_0$ and*

$$\mathcal{O}_R(1,0) \cong \mathcal{O}_R(e), \;\; \mathcal{O}_R(0,1) \cong \mathcal{O}_R(2f).$$

*Moreover a unique cubic Segre product contains R as an element of $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(0,2)|$.*

*Proof.* Let $p_2 : \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^2$ the second projection map. Since $|R| = p_2^*|\mathcal{O}_{\mathbb{P}^2}(2)|$, then $R = \mathbb{P}^1 \times B$, where $B \subset \mathbb{P}^2$ is a smooth conic. Moreover $\mathbb{P}^1 \times \mathbb{P}^2$ is the union of the planes $\{x\} \times \mathbb{P}^2, x \in \mathbb{P}^1$, and $\{x\} \times \mathbb{P}^2$ is spanned by the conic $\{x\} \times B \subset R$. Hence $\mathbb{P}^1 \times \mathbb{P}^2$ is uniquely associated to $R$. □

**Proposition 6.1.3.** *Let $R \in \left|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(2,1)\right|$ be a smooth and irreducible divisor. Then $R$ is $\mathbb{F}_n$ with $n \in \{0,2\}$ and*

$$\mathcal{O}_R(1,0) \cong \mathcal{O}_R(f), \ \ \mathcal{O}_R(0,1) \cong \mathcal{O}_R\left(\frac{n+2}{2}f + e\right).$$

*The family of cubic Segre products containing $R$ as an element of $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(2,1)|$ is naturally parametrized by an open set of $|\mathcal{O}_R(0,1)|^*$.*

*Proof.* We have that $p_2|_R : R \to \mathbb{P}^2$ is a generically finite morphism of degree 2. Fixing bi-homogeneous coordinates $[Z_0 : Z_1] \times [X_0 : X_1 : X_2]$ on $\mathbb{P}^1 \times \mathbb{P}^2$, the equation of $R$ is $aZ_0^2 + bZ_0Z_1 + cZ_1^2 = 0$, where $a, b, c$ are linear forms in $[X_0 : X_1 : X_2]$. Since $R$ is smooth, then the branch curve $B$ of $p_2|_R$ is a conic of rank $\geq 2$. If $B$ is smooth then $p_2|_R$ is finite and $R \cong \mathbb{F}_0$. If $B$ has rank 2, then $R$ is the blowing up of a quadric cone in its singular point, so it is $\mathbb{F}_2$. Now observe that $V_1 := p_1|_R^* \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(1)) = \mathrm{H}^0(\mathcal{O}_R(f))$ and that $\mathcal{L} := p_2|_R^*\mathcal{O}_{\mathbb{P}^2}(1)$ is the line bundle defining the model of $R$ as a quadric surface. in particular $V_2 := p_2|_R^* \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^2}(1))$ has codimension 1 in $\mathrm{H}^0(\mathcal{L})$. Finally consider

$$\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(1,0)) \otimes \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(0,1)) \xrightarrow{r} V_1 \otimes V_2 \xrightarrow{m} \mathrm{H}^0(\mathcal{O}_R(1))$$

where $r$ is the restriction and $m$ is the multiplication map. It is standard to check that both $r$ and $m$ are isomorphisms. This implies that $V_2$ uniquely reconstructs $\mathbb{P}^1 \times \mathbb{P}^2$ and that the family of cubic Segre products containing $R$ as an element of $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(2,1)|$ is birationally parametrized by $|\mathcal{O}_R(0,1)|^*$. $\square$

Let $R$ be a smooth quartic scroll, that is a Hirzebruch surface $\mathbb{F}_n$ with $n \in \{0,2\}$. Keeping in account the previous propositions and their proofs, it is easy to associate to $R$ a union of planes $T$ containing $R$ as follows. Consider the pencil $\left|f + \frac{n}{2}e\right|$ and the union of planes

$$T := \bigcup_{c \in \left|f+\frac{n}{2}e\right|} T_c$$

where $T_c$ is the plane spanned by $c$. Notice that $c$ is a smooth conic if $n = 0$ and the rank 2 conic $e + f', f' \in |f|$ if $n = 2$.

**Theorem 6.1.4.** *$R$ is in a unique cubic Segre product if $n = 0$ and in a unique cone of vertex $e$ over a rational normal cubic if $n = 2$.*

In what follows it will be enough to assume $n = 0$. We have a chain of embeddings

$$\mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{\begin{pmatrix} \mathrm{id}_{\mathbb{P}^1} & 0 \\ 0 & |\mathcal{O}(2)| \end{pmatrix}} \mathbb{P}^1 \times \mathbb{P}^2 \xrightarrow{|\mathcal{O}(1,1)|} \mathbb{P}^5$$

Let $F \in \mathrm{H}^0\left(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(0,2)\right)$ be the defining polynomial of $R$ in $T$. Then the restriction map

$$\pi : \mathrm{H}^0\left(\mathcal{O}_{\mathbb{P}^5}(3)\right) \to \mathrm{H}^0\left(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(3,3)\right)$$

has the property that $\pi(\mathrm{H}^0(\mathcal{I}_R(3))) = f \cdot \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(3,1))$. The restriction

$$\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(3,1)) \to \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(3,2))$$

is an isomorphism of vector spaces. Consequently there is an induced homomorphism

$$\mathrm{H}^0(\mathcal{I}_R(3)) \to \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(3,2))$$

which induces a linear projection.

$$\mathfrak{C}_R \to \left|\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(3,2)\right|$$

We want this map to descend to a $\mathbb{P}^{16}$-bundle

$$\tilde{\mathcal{C}}_{14} \to \mathrm{Pic}_{3,2}$$

Recall some general fact about GIT.

**Definition 6.1.5.** Let $X$ be and algebraic variety, let $G$ be a reductive algebraic group acting on $X$ and let $\mathcal{F}$ be a coherent sheaf on the semistable locus $X^{ss}$. Then $\mathcal{F}$ is said to descend to $X//G$ if there is a coherent sheaf $\overline{\mathcal{F}}$ on $X//G$ whose pullback under the quotient map $X^{ss} \to X//G$ is the original sheaf $\mathcal{F}$.

If $\mathcal{F}$ is a vector bundle the following result gives a criteria for descent:

**Theorem 6.1.6** (Kempf). *Let $X$ be a quasi-projective scheme over an algebraically closed field $\kappa$ of characteristic zero, and let $G$ be a reductive algebraic group defined over $\kappa$ which acts on $X$ with a fixed choice of linearization $H$. Let $E$ be a $G$-vector bundle on $X^{ss}$. Then $E$ descends to $X//G$ if and only if for every closed point $x$ of $X^{ss}$ such that the orbit $G \cdot x$ is closed in $X^{ss}$, the stabilizer of $x$ in $G$ acts trivially on the fiber $E_x$ of $E$ at $x$.*

*Proof.* See [DN]. □

Recall also a standard result ([Har2, 12.9]):

**Theorem 6.1.7** (Grauert)**.** *Let $f : X \to Y$ be a projective morphism of noetherian schemes, and let $\mathcal{F}$ be a coherent sheaf on $X$, flat over $Y$. Suppose furthermore that $Y$ is integral and that for some $i$, the function $h^i(X_y, \mathcal{F}_y)$ is constant on $Y$. Then $R^i\mathcal{F}$ is locally free on $Y$ and the natural map*

$$R^i f_*(\mathcal{F}) \otimes k(y) \to \mathrm{H}^i(X_y, \mathcal{F}_y)$$

*is an isomorphism.*

The Kempf and Grauert theorems are the key ingredients in the proof of the next proposition. A similar argument was used by Shepherd-Barron in [She, 6] to prove the rationality of $\mathcal{M}_6$.

**Proposition 6.1.8.** *$\tilde{\mathcal{C}}_{14}$ is birational to $\mathbb{P}^{16} \times \mathbb{P}_{3,2}$.*

*Proof.* The rational map

$$\mathfrak{C}_R \to |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(3,2)|$$

is a linear projection

$$\pi : \mathbb{P}(V) \to \mathbb{P}(V')$$

where $V := H^0(\mathcal{I}_R(3)), V' := H^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(3,2))$. Let $\alpha : \tilde{\mathbb{P}} \to \mathbb{P}(V)$ be the blow-up a of the base locus of the projection and $\tilde{\pi} := \alpha \circ \pi$. Let $\mathcal{L} := \alpha^* \mathcal{O}_{\mathbb{P}(V)}(1)$. We have that $\mathrm{PGL}(2)^2$ acts freely on a open subvariety of $\mathbb{P}(V')$, so $\mathcal{L}$ is $\mathrm{PGL}(2)^2$-linearized. It follows from Kempf theorem that it descends to a line bundle on $\mathbb{P}(V) /\!/ \mathrm{PGL}(2)^2$. Furthermore it restricts to $\mathcal{O}(1)$ on the fibers of the map

$$\tilde{\mathbb{P}} \to \mathbb{P}(V').$$

From Grauert theorem we have that $\tilde{\pi}_* \mathcal{L}$ is locally free on $\mathbb{P}(V')$ and the projective fibration $\alpha$ is isomorphic to $\mathbb{P}(\tilde{\pi}_* \mathcal{L})$. It follows that $\alpha$ is a projective bundle. □

## 6.2 Rationality of $\mathrm{Pic}_{3,2}$

We show that the projectivization of the pull-back on $\mathrm{Pic}_{3,2}$ of the Hodge bundle over $\mathcal{M}_2$ is a rational variety. It follows that $\mathrm{Pic}_{3,2} \times \mathbb{P}^1$ is rational. Then a fortiori also $\tilde{\mathcal{C}}_{14}$ is rational. Recall that $\mathcal{M}_g$ is endowed with a sheaf named *Hodge bundle*. Over a suitable non empty open set the Hodge bundle is a rank $g$ vector bundle $\Lambda_g$ with fibre $H^0(\omega_C)$ at the moduli point of C, see e.g. [Loo1] or [HM].

**Definition 6.2.1.** The Hodge bundle over $\mathrm{Pic}_{3,2}$ is the rank-2 vector bundle

$$\Lambda_{3,2} \to \mathrm{Pic}_{3,2}$$

defined as the pullback of $\Lambda_2 \to \mathcal{M}_2$ under the natural map $\mathrm{Pic}_{3,2} \to \mathcal{M}_2$

Let $\mathbb{K}_g$ be the projectivization of $\Lambda_g$. Then $\mathbb{K}_g$ fits in the general theory of moduli of abelian differentials, see [FP]. Indeed an open set of it is the coarse moduli space of couples $(C, K)$ such that $C$ is a smooth, connected genus $g$ curve and $K$ is a smooth canonical divisor of $C$.

**Definition 6.2.2.** We denote by $\mathbb{K}_{3,2}$ the pull-back of $\mathbb{K}_2$ through the natural map $\mathrm{Pic}_{3,2} \to \mathcal{M}_2$.

In particular it follows that $\mathbb{K}_{3,2}$ represents the coarse moduli space of triples $(C, \mathcal{L}, K)$. Let us consider the projection map

$$p : \mathbb{K}_{3,2} \to \mathrm{Pic}_{3,2}$$

then $p$ is a $\mathbb{P}^1$-bundle over an open set of $\mathrm{Pic}_{3,2}$. Its fibre over the moduli point of $(C, \mathcal{L})$ in $\mathrm{Pic}_{3,2}$ is $|\omega_C|$. Now it is not difficult to construct a family of triples $(C, \mathcal{L}, K)$ dominating $\mathbb{K}_{3,2}$ via the moduli map.

For a general triple $(C, \mathcal{L}, K)$ we can assume that $\mathcal{L}$ is globally generated and that $K$ consists of two distinct points $K = o_1 + o_2$ with $o_1 \neq o_2$. Let $p : C \to \mathbb{P}^1$ and $q := C \to \mathbb{P}^1$ be the morphisms respectively defined by $\omega_C$ and $\mathcal{L}$. Then $p \times q$ defines an embedding

$$C \subset \mathbb{P}^1 \times \mathbb{P}^1$$

with two marked points, that are the images of $o_1$ and $o_2$. With some abuse of notation, we still denote them by $o_1, o_2$. In particular $C$ is a smooth element of the linear system $|\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(3,2)|$ and contains $\{o_1, o_2\}$. et $\mathcal{I}$ be the ideal sheaf of $\{o_1, o_2\}$ in $\mathbb{P}^1 \times \mathbb{P}^1$ and let

$$\begin{aligned} |\mathcal{I}(3,2)| &\xrightarrow{\quad m \quad} \mathbb{K}_{3,2} \\ C &\longmapsto (C, \mathcal{O}_C(0,1), o_1 + o_2) \end{aligned}$$

be the natural moduli map. Since $(C, \mathcal{L}, K)$ defines a general point of $\mathbb{K}_{3,2}$ the next property is immediate.

**Proposition 6.2.3.** $m : |\mathcal{I}(3,2)| \dashrightarrow \mathbb{K}_{3,2}$ *is dominant.*

The following result gives a more concrete description of $\mathbb{K}_{3,2}$.

**Proposition 6.2.4.** *Let $m : |\mathcal{I}(3,2)| \dashrightarrow \mathbb{K}_{3,2}$ be as above. Then*

$$m(C) = m(C') \iff \exists \sigma \in \mathrm{Stab}_{\mathrm{PGL}(2)^2}(\{o_1, o_2\}) \text{ s.t. } \sigma(C) = C'$$

*Proof.* The isomorphisms $\sigma : C \to C'$ which are restrictions of an element of $\mathrm{Aut}(\mathbb{P}^1)^2$ are determined by the conditions

$$\sigma^* \mathcal{O}_{C'}(1,0) = \mathcal{O}_C(1,0), \;\; \sigma^* \mathcal{O}_{C'}(0,1) = \sigma^* \mathcal{O}_C(0,1)$$

Imposing that $\sigma^*(o_1 + o_2) = o_1 + o_2$ means exactly that $\sigma \in \mathrm{Stab}_{\mathrm{PGL}(2)^2}(\{o_1, o_2\})$. $\square$

**Observation 6.2.5.** Since it is possible to move any $\{o_1, o_2\} \subset D \in |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,0)|$ to $\{([1:0],[1:0]),([1:0],[0:1])\}$ through an element of $\mathrm{PGL}(2)^2$, $\mathbb{K}_{3,2}$ can be described as the quotient of

$$\{C \in \left|\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(3,2)\right| : ([1:0],[0:1]),([1:0],[1:0]) \in C\}$$

modulo the equivalence relation

$$\begin{gathered} C \sim C' \\ \iff \\ \exists \sigma = (\sigma_1, \sigma_2) : \sigma(C) = C, \sigma_1([1:0]) = [1:0], \\ \sigma_2(\{[1:0],[0:1]\}) = \{[1:0],[0:1]\} \end{gathered}$$

We use the characterization of Observation 6.2.5 to prove its rationality using an argument of classical GIT.

**Proposition 6.2.6.** $\mathbb{K}_{3,2}$ *is birational to* $\mathbb{P}^6$*.*

*Proof.* From Observation 6.2.5 $\mathbb{K}_{3,2}$ can be described as the GIT quotient set of $(3,2)$-divisors

$$\left\{ \begin{gathered} C_{011} X_0^3 Y_0 Y_1 + X_0^2 X_1 (C_{120} Y_0^2 + C_{111} Y_0 Y_1 + C_{102} Y_1^2) + \\ + X_0 X_1^2 (C_{220} Y_0^2 + C_{211} Y_0 Y_1 + C_{202} Y_1^2) + \\ + X_1^3 (C_{320} Y_0^2 + C_{311} Y_0 Y_1 + C_{302} Y_1^2) = 0 : C_{ijk} \in \mathbb{C} \end{gathered} \right\}$$

modulo the action of the group $G \subset \mathrm{PGL}(2)^2$ defined by

$$G := \mathrm{Stab}(\{o_1, o_2\}) = \left\{ \left( \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}, \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix} \right) \right\} \cup \left\{ \left( \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}, \begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix} \right) \right\}$$

where $o_1 := ([1:0],[1:0])$, $o_2 := ([1:0],[0:1])$. For a general element of this set we have that $C_{011} \neq 0$. It follows that in the $G$-orbit of a general element there is an element with $C_{111} = 0$: just apply the transformation $X_0 \mapsto X_0 + \frac{C_{111}}{3C_{011}} X_1$. Note that if two elements with $C_{111} = 0$ are in the same $G$-orbit, then they are necessarily connected by an element of $G' \leq G$, where

$$G' := \left\{ \left( \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}, \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix} \right) \right\} \cup \left\{ \left( \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}, \begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix} \right) \right\}$$

and vice versa $G'$ acts on set of polynomials with $C_{111} = 0$. So another birational model of $\mathbb{K}_{3,2}$ is the GIT quotient

$$\left\{ \begin{matrix} C_{011} X_0^3 Y_0 Y_1 + X_0^2 X_1 (C_{120} Y_0^2 + C_{102} Y_1^2) + \\ + X_0 X_1^2 (C_{220} Y_0^2 + C_{211} Y_0 Y_1 + C_{202} Y_1^2) + \\ + X_1^3 (C_{320} Y_0^2 + C_{311} Y_0 Y_1 + C_{302} Y_1^2) = 0 \end{matrix} \right\} /\!\!/ G'$$

The same space can be described as the GIT quotient

$$\left\{ \begin{matrix} C_{011} Y_0 Y_1 + x (C_{120} Y_0^2 + C_{102} Y_1^2) + \\ + x^2 (C_{220} Y_0^2 + C_{211} Y_0 Y_1 + C_{202} Y_1^2) + \\ + x^3 (C_{320} Y_0^2 + C_{311} Y_0 Y_1 + C_{302} Y_1^2) \end{matrix} \right\} /\!\!/ H$$

where $H := \mathbb{C}^{*3} \rtimes (\mathbb{Z}/2\mathbb{Z})$. $H$ acts in the following way:

- $\mathbb{C}^{*3}$ acts by multiplying the variables $x, Y_0, Y_1$ by costants, more precisely:
$$(a,b,c) \cdot (C_{ijk}) = (a^i b^j c^k C_{ijk});$$
- $\mathbb{Z}/2\mathbb{Z}$ inverts the variables $Y_0$ and $Y_1$, more precisely
$$(1 \mod 2) \cdot (C_{ijk}) = C_{ikj}.$$

We find now 6 invariants which completely determine the GIT quotient of the dense open subset

$$\left\{ \begin{matrix} C_{011} \neq 0, C_{120} \neq 0, C_{102} \neq 0, C_{220} \neq 0, C_{211} \neq 0, \\ C_{202} \neq 0, C_{320} \neq 0, C_{311} \neq 0, C_{302} \neq 0 \end{matrix} \right\} /\!\!/ H$$

then giving a birational map to $\mathbb{C}^6$. We first compute the invariants for the action of $\mathbb{C}^{*3}$, then we recover the invariants for the action of $H$. We use that

$$X /\!\!/ G \cong (X /\!\!/ N) /\!\!/ (G/N)$$

in our case $G = H, N = \mathbb{C}^{*3}, G/N = \mathbb{Z}/2\mathbb{Z}$. The action of $\mathbb{C}^{*3}$ is uniquely determined by the following invariants

$$(I_1, J_2, J_3, I_4, I_5, I_6)$$
$$=$$
$$\left(\frac{C_{120}C_{102}}{C_{211}C_{011}}, \frac{C_{011}C_{311}}{C_{220}C_{102}}, \frac{C_{302}C_{011}}{C_{211}C_{102}}, \frac{C_{220}C_{202}}{C_{211}^2}, \frac{C_{320}C_{302}}{C_{311}^2}, \frac{C_{211}^3}{C_{311}^2C_{011}}\right)$$

It is immediate that they are invariants, we need to check that they uniquely determine an isomorphism class. Suppose that two polynomials (identified with elements of $\mathbb{C}^{*9}$)
$(C_{011}, C_{120}, C_{102}, C_{220}, C_{211}, C_{202}, C_{320}, C_{311}, C_{302})$ and
$(C'_{011}, C'_{120}, C'_{102}, C'_{220}, C'_{211}, C'_{202}, C'_{320}, C'_{311}, C'_{302})$ have the property that

$$(I_1, J_2, J_3, I_4, I_5, I_6) = (I'_1, J'_2, J'_3, I'_4, I'_5, I'_6) \tag{6.1}$$

Acting by $\mathbb{C}^{*3}$ on both of them it is possible to make

$$C_{211} = C'_{211} = 1, \quad C_{311} = C'_{311} = 1, \quad C_{102} = C'_{102} = 1 \tag{6.2}$$

in fact it is just needed to choose two triples $(a, b, c)$ and $(a', b', c')$ such that

$$\begin{cases} a^2bc = C_{211}^{-1} \\ a^3bc = C_{311}^{-1} \\ ac^2 = C_{102}^{-1} \end{cases} \qquad \begin{cases} a'^2b'c' = C'^{-1}_{211} \\ a'^3b'c' = C'^{-1}_{311} \\ a'c'^2 = C'^{-1}_{102} \end{cases}$$

If the equalities 6.1 and 6.2 hold, then $C_{ijjk} = C'_{ijk}$, in fact:

1. $I_6 = I'_6 \iff \dfrac{C_{211}^3}{C_{311}^2C_{011}} = \dfrac{C'^3_{211}}{C'^2_{31}C'_{011}} \Rightarrow C_{011} = C'_{011}$;

2. $J_2 = J'_2 \iff \dfrac{C_{011}C_{311}}{C_{220}C_{102}} = \dfrac{C'_{011}C'_{311}}{C'_{220}C'_{102}} \Rightarrow C_{220} = C'_{220}$;

3. $I_4 = I'_4 \iff \dfrac{C_{220}C_{202}}{C_{211}^2} = \dfrac{C'_{220}C'_{202}}{C'^2_{211}} \Rightarrow C_{202} = C'_{202}$;

4. $I_1 = I'_1 \iff \dfrac{C_{120}C_{102}}{C_{211}C_{011}} = \dfrac{C'_{120}C'_{102}}{C'_{211}C'_{011}} \Rightarrow C_{120} = C'_{120}$;

5. $J_3 = J'_3 \iff \dfrac{C_{302}C_{011}}{C_{211}C_{102}} = \dfrac{C'_{302}C'_{011}}{C'_{211}C'_{102}} \Rightarrow C_{302} = C'_{320}$;

6. $I_5 = I'_5 \Longleftrightarrow \dfrac{C_{320}C_{302}}{C_{311}^2} = \dfrac{C'_{320}C'_{302}}{C'^2_{311}} \Rightarrow C_{320} = C'_{320}$.

Note that the invariants $J_2, J_3$ are not invariant for the action of $\mathbb{Z}/2\mathbb{Z}$. In fact let $\iota$ be the involution exchanging the variables $Y_0$ and $Y_1$, then

$$\iota(J_2) = \frac{C_{011}C_{311}}{C_{202}C_{120}} = I_1^{-1}J_2{}^{-1}I_4^{-1}I_6^{-1}$$
$$\iota(J_3) = \frac{C_{320}C_{011}}{C_{211}C_{120}} = I_1^{-1}J_3{}^{-1}I_5I_6^{-1}$$

It is standard to check that the action of $\mathbb{Z}/2\mathbb{Z}$ on $\mathbb{C}^{*6}$ given by

$$\iota(x_1, x_2, x_3, x_4, x_5, x_6) = (x_1, x_1^{-1}x_2^{-1}x_4^{-1}x_6^{-1}, x_1^{-1}x_3^{-1}x_5x_6^{-1}, x_4, x_5, x_6)$$

is uniquely determined by the invariants

$$(x_1, x_2 + x_1^{-1}x_2^{-1}x_4^{-1}x_6^{-1}, x_3 + x_1^{-1}x_3^{-1}x_5x_6^{-1}, x_4, x_5, x_6)$$

It follows that the invariants of the action of $H$ are

$$(I_1, I_2, I_3, I_4, I_5, I_6)$$
$$=$$
$$\left(\frac{C_{120}C_{102}}{C_{211}C_{011}}, \frac{C_{011}C_{311}}{C_{220}C_{102}} + \frac{C_{011}C_{311}}{C_{202}C_{120}}, \frac{C_{302}C_{011}}{C_{211}C_{102}} + \frac{C_{320}C_{011}}{C_{211}C_{120}}, \frac{C_{220}C_{202}}{C_{211}^2}, \frac{C_{320}C_{302}}{C_{311}^2}, \frac{C_{211}^3}{C_{311}^2C_{011}}\right)$$

which gives a birational correspondence between $\mathbb{K}_{3,2}$ and $\mathbb{C}^6$. □

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